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Cartan forms for first order constrained variational problems

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Abstract

Given a constrained variational problem on the 1-jet extension J^1Y of a fibre bundle $p : Y \to X$, under certain conditions on the constraint submanifold $S \subset J^1Y$, we characterize the space of admissible infinitesimal variations of an admissible section $s : X \to Y$ as the image by a certain first order differential operator, P_s, of the space of sections $\Gamma(X, s^*VY)$. In this way we obtain a constrained first variation formula for the Lagrangian density $\mathcal{L}\omega$ on J^1Y, which allows us to characterize critical sections of the problem as admissible sections s such that $P_s^+\mathcal{E}_{\mathcal{L}\omega}(s) = 0$, where P_s^+ is the adjoint operator of P_s and $\mathcal{E}_{\mathcal{L}\omega}(s)$ is the Euler–Lagrange operator of the Lagrangian density $\mathcal{L}\omega$ as an unconstrained variational problem. We introduce a Cartan form on J^2Y which we use to generalize the Cartan formalism and Noether theory of infinitesimal symmetries to the constrained variational problems under consideration. We study the relation of this theory with the Lagrange multiplier rule as well as the question of regularity in this framework. The theory is illustrated with several examples of geometrical and physical interest.

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1. Introduction

The problem of Lagrange in the calculus of variations on fibred manifolds can be established as follows:

Let $\mathcal{L}\omega$ be a Lagrangian density on the 1-jet extension J^1Y of a fibre bundle $p : Y \to X$ on an n-dimensional oriented manifold X ($\mathcal{L} \in \mathcal{C}^\infty(J^1Y)$ and ω a volume element on X) and let S be a submanifold of J^1Y such that $(j^1p)(S) = X$ (the constraint). A section s is said to be *admissible* if $\mathrm{Im}(j^1s) \subset S$, and given an admissible section s an *admissible infinitesimal variation* of s is a p-vertical vector field along s, $D_s^{\mathrm{v}} \in \Gamma(X, s^*VY)$ whose 1-jet extension $j^1D_s^{\mathrm{v}}$ is tangential to the submanifold S along j^1s. The Lagrangian density defines on the set $\Gamma_S(X, Y)$ of admissible sections the functional $\mathbb{L}(s) = \int_{j^1s} \mathcal{L}\omega$ and, given $s \in \Gamma_S(X, Y)$, if $T_s(\Gamma_S(X, Y))$ is the vector space of admissible infinitesimal variations of s, the differential of $\mathbb{L}$ at s is defined as $(\delta_s\mathbb{L})(D_s^{\mathrm{v}}) = \int_{j^1s} L_{j^1D_s^{\mathrm{v}}}(\mathcal{L}\omega)$, for $D_s^{\mathrm{v}} \in T_s(\Gamma_S(X, Y))$. In this situation, an admissible section $s \in \Gamma_S(X, Y)$ is said to be *critical* for the constrained variational problem defined by the pair $(\mathcal{L}\omega, S \subset J^1Y)$ when $\delta_s\mathbb{L} = 0$ on the subspace $T_s^{\mathrm{c}}(\Gamma_S(X, Y))$ of infinitesimal admissible variations with compact support; the main objective of the problem of Lagrange is to determine these critical sections.

The traditional method to solve this problem has been the so-called *Lagrange multiplier rule*, which assumes that the constraint submanifold may be expressed in the form $S = \{j_x^1s \in J^1Y/\Phi(j_x^1s) = 0\}$, where $q : E \to Y$ is a vector bundle on Y and $\Phi : J^1Y \to E$ is a bundle morphism on E satisfying certain regularity conditions (Hypothesis (HY1) of Section 3).

Assuming this, if one considers the unconstrained variational problem on $J^1(Y \times_Y E^*)$ (where E^* is the dual bundle of E) of Lagrangian density $(\mathcal{L} + \lambda \circ \Phi)\omega$, where $\lambda \in \Gamma(J^1(Y \times_Y E^*), E^*)$ is the tautological section $\lambda(j_x^1(s, \lambda)) = \lambda(x)$ and $\circ$ denotes the bilinear duality product, it holds that if $(s, \lambda) \in \Gamma(X, Y \times_Y E^*)$ is critical for the unconstrained variational problem $(\mathcal{L} + \lambda \circ \Phi)\omega$ then $s \in \Gamma(X, Y)$ is critical for the constrained variational problem $(\mathcal{L}\omega, S \subset J^1Y)$.

In this way one may define a mapping:

$$\Pi : \Gamma_{\mathrm{crit}}(X, Y \times_Y E^*) \to \Gamma_{\mathrm{crit}}(X, Y)$$

from the set of critical sections of the unconstrained variational problem $(\mathcal{L} + \lambda \circ \Phi)\omega$ to the set of critical sections of the constrained variational problem $(\mathcal{L}\omega, S \subset J^1Y)$, that need not be injective nor surjective. A fundamental question of the problem of Lagrange is to determine sufficient conditions which allow us to ensure the surjectivity of Π.

In this paper we prove that, by imposing another condition on the constraint submanifold S (Hypothesis (HY2) of Section 3), it is possible to invert the mapping Π on the sections $s \in \Gamma_{\mathrm{crit}}(X, Y)$ such that $\mathrm{Im}\, j^1s$ belongs to a certain dense open subset of S, which solves the problem of Lagrange for this class of constrained variational problems in the best possible way. Moreover, this method allows us to associate to these problems a Cartan form with which we extend the corresponding Hamilton–Cartan formalism and Noether theory of infinitesimal symmetries from the unconstrained variational calculus, which has been so intensely studied in the literature (see, for example [1,4,9,12–15,18–20,24,26,28–30,33,36,37] and references therein). In particular, in the case of one independent variable, one recovers from this general viewpoint the also abundant literature on the problem of Lagrange, specially that of the last years, whose physical and geometrical interest is well known (vakonomic mechanics, subriemannian geometry, Morse theory for constrained problems, etc., see for example [2,3,5,21–23,25,34] and references therein).

The interest of the subject has recently increased even more due to the new approach to the problem of Lagrangian reduction, according to which a certain kind of variational problems, called *reducible*, can be *reduced* to constrained variational problems of a lower order, which serves as motivation to take these last problems together with the associated structures as *objects* of a possible *variational category* which includes the Lagrangian reduction procedure as one of its fundamental operations (see, for example [6–8,10,11,31,35]).

The plan of the work is as follows: After a brief review of the unconstrained case in Section 2, where we fix the method and notation, Section 3 constitutes the core of our approach where we establish Hypotheses (HY1) and (HY2) on the constraint submanifold $S \subset J^1Y$, which allow us to characterize the space $T_s(\Gamma_S(X, Y))$ of infinitesimal admissible variations of an admissible section $s \in \Gamma_S(X, Y)$ as the image of certain first order linear differential operator, P_s, on the space of sections $\Gamma(X, s^*VY)$ (Theorem 3.7). This is indeed the main result of this section, and allows us to obtain a very nice constrained first variation formula for the Lagrangian density (Theorem 3.8), with which we may characterize the critical sections of the constrained variational problem as those admissible sections $s \in \Gamma_S(X, Y)$ such that $P_s^+ \mathcal{E}_{\mathcal{L}\omega}(s) = 0$, where P_s^+ is the adjoint operator of P_s and $\mathcal{E}_{\mathcal{L}\omega}$ is the Euler–Lagrange operator of $\mathcal{L}\omega$ as an unconstrained variational problem (Corollary 3.9).

In Section 4 we introduce the Cartan form $\widetilde{\Theta}$ on J^2Y (Definition 4.1) which allows us to characterize critical sections by means of the corresponding Cartan equation (Theorem 4.3), and to generalize Noether theory of infinitesimal symmetries to the constrained variational problems under consideration (Definition 4.4 and Theorem 4.5).

In Section 5 we study the relation with the Lagrange multiplier rule, proving the fundamental bijection $\Pi : \Gamma_{\text{crit}}(X, Y \times_Y E^*) \xrightarrow{\sim} \Gamma_{\text{crit}}(X, Y)$. Delving deeper into this relation, it is also proven that the Cartan forms $\Theta_{(\mathcal{L}+\lambda\circ\Phi)\omega}$ and $\widetilde{\Theta}$ of both problems can be projected on $J^1Y \times_Y E^*$ to a common n-form $\widehat{\Theta}$, so that the corresponding Cartan and Noether formalisms can be reduced to this fibred manifold (Theorem 5.1 and Proposition 5.2). Moreover, taking the restriction of $\widehat{\Theta}$ to the submanifold $S \times_Y E^* \subset J^1Y \times_Y E^*$ it is then possible to state a notion of regularity (Definition 5.3), which allows us to identify critical sections of the constrained variational problem with those sections $\widehat{s} = (\overline{s}, \lambda) \in \Gamma(X, S \times_Y E^*)$ that satisfy the Cartan equation $\widehat{s}^* i_{\widehat{D}} \mathrm{d}\widehat{\Theta} = 0$, for any $\widehat{D} \in \mathfrak{X}(S \times_Y E^*)$ (Theorem 5.5).

In Section 6 the whole theory is illustrated with some classical examples in one or more independent variables, with emphasis on the validity of Hypothesis (HY2), which constitutes the basis of our approach.

The work finishes with an Appendix where we prove that the main results of the theory do not depend on the chosen vector bundle $q : E \to Y$ nor on the bundle morphism $\Phi : J^1Y \to E$ that define the constraint submanifold $S = \Phi^{-1}(0_E)$.

All manifolds, mappings, tensors, etc. will be considered to be $\mathcal{C}^\infty$. The notion of fibre bundle will be understood in an ample sense, that is, a $\mathcal{C}^\infty$ locally trivial surjective submersion $p : Y \to X$. Throughout the paper we will use differential calculus with values in vector bundles, following the reference [27] without explicitly mentioning it.

2. A review of unconstrained variational calculus

Here, we summarize some aspects of first order unconstrained variational calculus that we shall use. For the purposes of this paper we shall follow the formulation developed in [13]. For other approaches to this topic, see [19,20,26,28] and references therein.

Let $p : Y \to X$ be a fibre bundle over a n-dimensional manifold X, oriented by a volume element ω. Let J^1Y be the 1-jet bundle of local sections of p, and $j^1p : J^1Y \to X$ and $\pi : J^1Y \to Y$ be the canonical projections. If $\dim Y = n + m$ and (x^ν, y^j), $1 \le \nu \le n$, $1 \le j \le m$ is a fibred local coordinate system for p, we shall denote by (x^ν, y^j, y^j_ν) the natural induced coordinate system for J^1Y, where $y^j_\nu(j^1_x s) = ((\partial/\partial x^\nu)(y^j \circ s))(x)$ for any section $s \in \Gamma(X, Y)$.

Definition 2.1. Given a section $s \in \Gamma(X, Y)$, the vertical differential of s at a point $x \in X$ is the linear mapping $(\mathrm{d}^{\mathrm{v}} s)_x : T_{s(x)}Y \to V_{s(x)}Y$ given by the formula:

$$(\mathrm{d}^{\mathrm{v}} s)_x D_{s(x)} = D_{s(x)} - (s \circ p)_* D_{s(x)}, \quad D_{s(x)} \in T_{s(x)}Y$$

where VY is the vertical bundle of the projection p.

This notion allows us to define a 1-form θ on J^1Y with values in the induced vector bundle VY_{J^1Y}, by the rule:

$$\theta_{j^1_x s}(D_{j^1_x s}) = (\mathrm{d}^{\mathrm{v}} s)_x(\pi_* D_{j^1_x s}), \quad D_{j^1_x s} \in T_{j^1_x s}(J^1Y)$$

This is the so called structure 1-form of J^1Y, which is locally given by the expression:

$$\theta = \sum_j \left(\mathrm{d}y^j - \sum_\nu y^j_\nu \, \mathrm{d}x^\nu \right) \otimes \frac{\partial}{\partial y^j}$$

This 1-form defines the basic structure of J^1Y, with which the different notions on 1-jet bundles are characterized. For example:

A section $\bar{s} \in \Gamma(X, J^1Y)$ is the 1-jet extension of a section $s \in \Gamma(X, Y)$ (i.e. $\bar{s} = j^1 s$) if and only if $\bar{s}^*\theta = 0$. An infinitesimal contact transformation (i.c.t.) is a vector field $\bar{D}$ on J^1Y such that for any linear connection ∇ on VY there exists an endomorphism f of the induced vector bundle VY_{J^1Y} such that $L_{\bar{D}}\theta = f \circ \theta$, where the Lie derivative is taken with respect to the corresponding induced connection and the product "$\circ$" is the obvious one. This condition does not depend on the connection ∇ and it holds that for every vector field on Y (not necessarily p-projectable) there exists a unique i.c.t. j^1D projectable to D. Moreover, the map $D \mapsto j^1D$ is an injection of Lie algebras. The vector field j^1D is called the 1-jet extension of the vector field D. Locally, if $D = \sum_\nu u^\nu(\partial/\partial x^\nu) + \sum_j v^j(\partial/\partial y^j)$ $(u^\nu, v^j \in \mathcal{C}^\infty(Y))$, then its 1-jet extension is: $j^1D = \sum_\nu u^\nu(\partial/\partial x^\nu) + \sum_j v^j(\partial/\partial y^j) + \sum_{j\nu} w^j_\nu(\partial/\partial y^j_\nu)$

where:

$$w_\nu^j = \frac{\partial v^j}{\partial x^\nu} + \sum_k y_\nu^k \frac{\partial v^j}{\partial y^k} - \sum_\mu y_\mu^j \left(\frac{\partial u^\mu}{\partial x^\nu} + \sum_k y_\nu^k \frac{\partial u^\mu}{\partial y^k} \right)$$

In what follows we shall denote by $\mathfrak{X}^{(1)}(Y)$ the Lie algebra of all the i.c.t.s and by $\mathfrak{X}_c^{(1)}(Y)$ the ideal of this Lie algebra defined by the i.c.t.s whose supports have compact image in X.

A first order variational problem on the bundle $p : Y \to X$ is defined by a function $\mathcal{L} \in \mathcal{C}^\infty(J^1Y)$ (the Lagrangian). The n-form $\mathcal{L}\omega$ (Lagrangian density) defines then a functional $\mathbb{L} : \Gamma(X, Y) \to \mathbb{R}$ by the rule:

$$\mathbb{L}(s) = \int_{j^1 s} \mathcal{L}\omega = \int_X (j^1 s)^* \mathcal{L}\omega, \quad s \in \Gamma(X, Y)$$

where $\mathbb{L}$ is defined on the sections for which the above integral exists.

For each section $s \in \Gamma(X, Y)$ we define a linear form $\delta_s \mathbb{L} : \mathfrak{X}^{(1)}(Y) \to \mathbb{R}$ by the rule:

$$(\delta_s \mathbb{L})(\bar{D}) = \int_{j^1 s} L_{\bar{D}}(\mathcal{L}\omega), \quad \bar{D} \in \mathfrak{X}^{(1)}(Y) \tag{2.1}$$

Definition 2.2. A section s is critical for the Lagrangian density $\mathcal{L}\omega$ when $\delta_s \mathbb{L} = 0$ on $\mathfrak{X}_c^{(1)}(Y)$.

Similar treatments can be given for fixed boundary problems and other situations.

A central problem of the variational calculus is the characterization of critical sections as solutions of some differential system defined on a suitable jet bundle. The notion of Cartan form associated to a Lagrangian density not only solves this problem, but also allows us to generalize many notions from analytical mechanics to variational calculus. With the approach that we shall follow here, this basic concept can be introduced as follows:

Proposition 2.3 (Momentum form). *There exists a unique $VY^*_{J^1Y}$-valued $(n-1)$-form $\Omega_{\mathcal{L}\omega}$ on J^1Y such that $\Omega_{\mathcal{L}\omega} = i_F \omega$, where F is any $VY^*_{J^1Y}$-valued vector field on J^1Y, solution of the equation*:

$$i_F \, \mathrm{d}\theta = \mathrm{d}\mathcal{L}$$

over the π-vertical vector fields of J^1Y, where the exterior derivative is taken with respect to the induced connection on VY_{J^1Y} of a linear connection ∇ on VY, and the bilinear products are the obvious ones. The $(n-1)$-form $\Omega_{\mathcal{L}\omega}$ does not depend on the choice of the connection ∇.

Proof. Let (x^ν, y^j, y_ν^j) be a local coordinate system for J^1Y and let $\Gamma_{\nu i}^k$, $\bar{\Gamma}_{ji}^k$ be the coefficients of the connection ∇ on VY with respect to the coordinates (x^ν, y^j), that is:

$$\nabla_{\partial/\partial x^\nu} \left(\frac{\partial}{\partial y^i} \right) = \sum_k \Gamma_{\nu i}^k \left(\frac{\partial}{\partial y^k} \right), \quad \nabla_{\partial/\partial y^j} \left(\frac{\partial}{\partial y^i} \right) = \sum_k \bar{\Gamma}_{ji}^k \left(\frac{\partial}{\partial y^k} \right)$$

From here, using these local expressions and imposing the conditions of our statement in local coordinates, $\Omega_{\mathcal{L}\omega}$ is univocally determined by:

$$\Omega_{\mathcal{L}\omega} = \sum_{j,\nu} \frac{\partial \mathcal{L}}{\partial y_\nu^j} \omega_\nu \otimes \mathrm{d}y^j, \quad \omega_\nu = i_{(\partial/\partial x^\nu)}\omega \tag{2.2}$$

therefore, in virtue of the uniqueness of these local expressions, we conclude. □

The Cartan form associated to the Lagrangian density $\mathcal{L}\omega$ is now defined as the n-form on J^1Y:

$$\Theta_{\mathcal{L}\omega} = \theta \overline{\wedge} \Omega_{\mathcal{L}\omega} + \mathcal{L}\omega \tag{2.3}$$

where the exterior product $\overline{\wedge}$ is taken with respect to the bilinear product defined by duality.

This differential form has the following important property:

Proposition 2.4. *There exists an unique $VY^*_{J^1Y}$-valued (j^1p)-horizontal n-form $\mathcal{F}_{\mathcal{L}\omega}$ on J^1Y such that*:

$$\mathrm{d}\Theta_{\mathcal{L}\omega} = \theta \overline{\wedge} (\mathcal{F}_{\mathcal{L}\omega} - \mathrm{d}\Omega_{\mathcal{L}\omega}) \tag{2.4}$$

where the exterior derivative in the second member is taken with respect to the connection on J^1Y induced by a linear connection ∇ on VY with vanishing vertical torsion (i.e. $\nabla_{D_1} D_2 - \nabla_{D_2} D_1 - [D_1, D_2] = 0$ *for any pair D_1, D_2 of p-vertical vector fields on Y*).

Proof. In the local coordinate system from the proof of Proposition 2.3, the condition of vanishing vertical torsion leads to $\overline{\Gamma}_{ij}^k = \overline{\Gamma}_{ji}^k$. Taking this into account, Eq. (2.4) in the statement locally allows us to univocally determine $\mathcal{F}_{\mathcal{L}\omega}$ by the formula:

$$\mathcal{F}_{\mathcal{L}\omega} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial y^i} - \sum_{\nu,j} \left(\Gamma_{\nu i}^j + \sum_k y_\nu^k \overline{\Gamma}_{ki}^j \right) \frac{\partial \mathcal{L}}{\partial y_\nu^j} \right] \omega \otimes \mathrm{d}y^i$$

and again in virtue of uniqueness, we conclude. □

The $VY^*_{J^1Y}$-valued n-form $\mathbb{E}_{\mathcal{L}\omega} = \mathcal{F}_{\mathcal{L}\omega} - \mathrm{d}\Omega_{\mathcal{L}\omega}$ on J^1Y is called Euler–Lagrange form of the variational problem, and allows us to define the classical Euler–Lagrange operator $\mathcal{E}_{\mathcal{L}\omega} : s \in \Gamma(X, Y) \mapsto \mathcal{E}_{\mathcal{L}\omega}(s) \in \Gamma(X, s^*VY^*)$ by:

$$\mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega = (j^1 s)^* \mathbb{E}_{\mathcal{L}\omega}$$

In a local coordinate system, this operator has the well-known form:

$$\mathcal{E}_{\mathcal{L}\omega}(s) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial y^i} \circ j^1 s - \sum_\nu \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathcal{L}}{\partial y_\nu^i} \circ j^1 s \right) \right) \mathrm{d}y^i$$

Formula (2.4) is therefore a fundamental formula relating the three basic objects of the theory, that is: the structure form on the bundle of 1-jets, the Euler–Lagrange operator, and the Cartan form. Together with (2.3), it constitutes an intrinsic expression of the "Lepage congruences" from the classical calculus of variations [30]. From them, the whole theory can now be developed as follows:

Taking the Lie derivative of (2.3) with respect to any i.c.t. $\bar{D} \in \mathfrak{X}^{(1)}(Y)$ and bearing in mind (2.4) we have:

Theorem 2.5 (First variation formula). *There exists a* $VY^*_{J^1Y}$*-valued* $(n-1)$*-form* ξ *on* J^1Y (*depending on* $\bar{D}$) *such that*:

$$L_{\bar{D}}(\mathcal{L}\omega) = \theta(\bar{D}) \circ \mathbb{E}_{\mathcal{L}\omega} + \mathrm{d}(i_{\bar{D}}\Theta_{\mathcal{L}\omega}) + \theta \bar{\wedge} \xi, \quad \bar{D} \in \mathfrak{X}^{(1)}(Y) \tag{2.5}$$

We can now express the linear functional $\delta_s\mathbb{L}$ defined by (2.1) with the formula:

$$\begin{aligned}(\delta_s\mathbb{L})(\bar{D}) &= \int_{j^1s} L_{\bar{D}}(\mathcal{L}\omega) = \int_{j^1s} \theta(\bar{D}) \circ \mathbb{E}_{\mathcal{L}\omega} + \mathrm{d}(i_{\bar{D}}\Theta_{\mathcal{L}\omega}) \\ &= \int_X \mathcal{E}_{\mathcal{L}\omega}(s)(D^{\mathrm{v}}_s)\omega + \mathrm{d}\left(i_{\bar{D}}\Theta_{\mathcal{L}\omega}\right), \quad \bar{D} \in \mathfrak{X}^{(1)}(Y)\end{aligned} \tag{2.6}$$

where D^{v}_s is the vertical component along s of the projection D_s onto Y of the vector field $\bar{D}_{j^1s}$ along j^1s.

Due to formula (2.6) and to the fact that the mapping $\bar{D} \in \mathfrak{X}^{(1)}(Y) \mapsto D^{\mathrm{v}}_s \in \Gamma(X, s^*VY)$ is surjective, we may redefine the linear form $\delta_s\mathbb{L}$ on the "tangent space" $\Gamma(X, s^*VY)$ at $s \in \Gamma(X, Y)$ of the set of sections $\Gamma(X, Y)$ by the following formula:

$$(\delta_s\mathbb{L})(D^{\mathrm{v}}_s) = \int_{j^1s} L_{\bar{D}}(\mathcal{L}\omega) = \int_X \mathcal{E}_{\mathcal{L}\omega}(s)(D^{\mathrm{v}}_s)\omega + \mathrm{d}(\Omega_{\mathcal{L}\omega}(s)(D^{\mathrm{v}}_s)) \tag{2.7}$$

for $D^{\mathrm{v}}_s \in \Gamma(X, s^*VY)$, where $\bar{D} \in \mathfrak{X}^{(1)}(Y)$ is any i.c.t. extending D^{v}_s and $\Omega_{\mathcal{L}\omega}(s) = (j^1s)^*\Omega_{\mathcal{L}\omega}$.

In particular, for the sections with compact support $\Gamma^{\mathrm{c}}(X, s^*VY)$, one gets by Stokes' Theorem:

$$(\delta_s\mathbb{L})(D^{\mathrm{v}}_s) = \int_X \mathcal{E}_{\mathcal{L}\omega}(s)(D^{\mathrm{v}}_s)\omega, \quad D^{\mathrm{v}}_s \in \Gamma^{\mathrm{c}}(X, s^*VY)$$

therefore, by Definition 2.2 of critical section, we obtain:

Corollary 2.6 (Euler–Lagrange equation). *A section* $s \in \Gamma(X, Y)$ *is critical if and only if*:

$$\mathcal{E}_{\mathcal{L}\omega}(s) = 0$$

On the other hand, from (2.4) and the previous corollary follows:

Corollary 2.7 (Cartan equation). *A section* $s \in \Gamma(X, Y)$ *is critical if and only if for every vector field* D *on* J^1Y *it holds*:

$$(j^1s)^*\left(i_D\,\mathrm{d}\Theta_{\mathcal{L}\omega}\right) = 0$$

Following this guideline, Noether theory of infinitesimal symmetries of a variational problem can now be established as follows:

Definition 2.8. An infinitesimal symmetry of a variational problem with Lagrangian density $\mathcal{L}\omega$ on J^1Y is any vector field $D \in \mathfrak{X}(Y)$ such that $L_{j^1D}(\mathcal{L}\omega) = 0$.

From formula (2.3) and (2.4), again, and from the second characterization of critical sections (Corollary 2.7), follows:

Theorem 2.9 (Noether). *If D is an infinitesimal symmetry and s is a critical section of a variational problem with Lagrangian density $\mathcal{L}\omega$* on J^1Y, *then*:

$$\mathrm{d}[(j^1s)^* i_{j^1D}\Theta_{\mathcal{L}\omega}] = 0$$

The $(n-1)$-form $i_{j^1D}\Theta_{\mathcal{L}\omega}$ on J^1Y is called the Noether invariant corresponding to the infinitesimal symmetry D.

If we denote by $\mathrm{Sym}(\mathcal{L}\omega)$ the real Lie algebra of infinitesimal symmetries of the variational problem, this correspondence between infinitesimal symmetries and their Noether invariants allows us to introduce the notion of multi-momentum map as follows:

Definition 2.10. The multi-momentum map associated to the variational problem with Lagrangian density $\mathcal{L}\omega$ on J^1Y is the mapping $\mu_{\mathcal{L}\omega} : \Gamma(X, Y) \to \mathrm{Sym}(\mathcal{L}\omega)^* \otimes \Gamma(X, \Lambda^{n-1}T^*X)$ defined by the rule:

$$\mu_{\mathcal{L}\omega}(s)(D) = (j^1s)^* i_{j^1D}\Theta_{\mathcal{L}\omega}, \quad D \in \mathrm{Sym}(\mathcal{L}\omega)$$

It is important to note that this formulation of first order variational calculus has been generalized to higher order in [14,15,33]. A more recent review on this approach can be found in [9, Section 2]. For other approaches to this topic, see [12,24,37] and references therein.

3. First order constrained variational problems. First variation formula. Euler–Lagrange equations

Given a variational problem with Lagrangian density $\mathcal{L}\omega$ on the 1-jet bundle J^1Y of a bundle $p : Y \to X$, the additional data needed to define a first order constrained variational problem is a submanifold $S \subset J^1Y$ such that $(j^1p)(S) = X$ (the constraint). This kind of variational problems, first proposed and studied by Lagrange for the case of one independent variable, can be stated as follows:

Definition 3.1. A section $s \in \Gamma(X, Y)$ is said to be admissible if $\mathrm{Im}\, j^1s \subset S$.

This condition defines a system of first order partial differential equations for the section $s \in \Gamma(X, Y)$, whose set of solutions $\Gamma_S(X, Y)$ can be seen as some kind of “manifold”, for which the following notion of “tangent space” can be given:

Definition 3.2. Given an admissible section $s \in \Gamma_S(X, Y)$, an admissible infinitesimal variation of s is a p-vertical vector field along s, $D^{\mathrm{v}}_s \in \Gamma(X, s^*VY)$, whose 1-jet extension $j^1 D^{\mathrm{v}}_s = j^1 D^{\mathrm{v}}|_{j^1 s}$ (D^{v} any p-vertical extension of D^{v}_s to a neighborhood of s in Y) is tangential to the submanifold $S \subset J^1 Y$ along $j^1 s$.

This tangency condition defines a system of linear first order partial differential equations for the section $D^{\mathrm{v}}_s \in \Gamma(X, s^*VY)$ that can be seen as the "linearization" of the equation $\mathrm{Im}\, j^1 s \subset S$ at the solution $s \in \Gamma_S(X, Y)$. The real vector space $T_s(\Gamma_S(X, Y))$ defined by its solutions can be interpreted as the "tangent space" to the "manifold" $\Gamma_S(X, Y)$ at the point $s \in \Gamma_S(X, Y)$. In particular, we shall denote by $T^{\mathrm{c}}_s(\Gamma_S(X, Y))$ the subspace of sections in $T_s(\Gamma_S(X, Y))$ with compact support.

Remark. Given $s \in \Gamma_S(X, Y)$, if $\{s_t\}$ $(t \in (-\epsilon, \epsilon) \subset \mathbb{R}, \epsilon > 0)$ is a differentiable 1-parametric deformation of $s = s_0$ by sections of $\Gamma_S(X, Y)$, it is easy to see that the vector field $\left.\frac{\partial s_t}{\partial t}\right|_{t=0}$ belongs to $T_s(\Gamma_S(X, Y))$, but the converse does not necessarily hold. A classical problem in the calculus of variations with constraints is to determine the admissible sections for which such a result holds, which are called regular solutions of the equation $\mathrm{Im}\, j^1 s \subset S$ (see [25] for a recent treatment of this question for one independent variable). In any case, the present trend, which we shall follow here, is to take $T_s(\Gamma_S(X, Y))$ as the space of admissible infinitesimal variations for the problem of Lagrange, which, on the other hand, constitutes the basic principle of the so called "vakonomic method" developed for mechanical systems with non-holonomic constraints [2,5,21,34].

At this point, the Lagrangian density $\mathcal{L}\omega$ defines on the set $\Gamma_S(X, Y)$ of admissible sections the functional:

$$\mathbb{L}(s) = \int_{j^1 s} \mathcal{L}\omega, \quad s \in \Gamma_S(X, Y)$$

and the differential of $\mathbb{L}$ at any section $s \in \Gamma_S(X, Y)$:

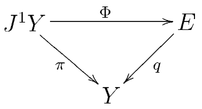
$$(\delta_s \mathbb{L})(D^{\mathrm{v}}_s) = \int_{j^1 s} L_{j^1 D^{\mathrm{v}}_s}(\mathcal{L}\omega), \quad D^{\mathrm{v}}_s \in T_s(\Gamma_S(X, Y))$$

Definition 3.3. An admissible section $s \in \Gamma_S(X, Y)$ is critical for the constrained variational problem with Lagrangian density $\mathcal{L}\omega$ on $J^1 Y$ and constraint submanifold $S \subset J^1 Y$ when $\delta_s \mathbb{L} = 0$ on the space $T^{\mathrm{c}}_s(\Gamma_S(X, Y))$ of admissible infinitesimal variations with compact support.

As for the case of unconstrained variational problems, a fundamental question now is the characterization of critical sections as solutions of some kind of partial differential equations. Under certain hypotheses on the constraint submanifold, in this work we shall give such a characterization obtaining an explicit Euler–Lagrange operator in an appropriate jet bundle. In addition, the procedure we follow allows us to construct a Cartan form for this kind of problems together with the subsequent generalization for this case of the corresponding Cartan formalism.

To specify the class of constraint submanifolds that we shall consider in the following, together with the first condition $(j^1 p)(S) = X$, we shall assume the following:

Hypothesis (HY1). There exists a rank k vector bundle $q : E \to Y$ and a bundle morphism:



such that $S = \Phi^{-1}(0_E)$ (where 0_E is the zero section of E) and the restriction of dΦ to the fibers of $\pi : J^1Y \to Y$ has rank k along S.

Locally, if $U \subset Y$ is an (x^μ, y^j)-coordinated open subset where E is trivial ($E_U = U \times \mathbb{R}^k$), and e_α ($\alpha = 1, \ldots, k$) is a local basis for the module of sections $\Gamma(U, E|_U)$ induced by this trivialization $E_U = U \times \mathbb{R}^k$, then $\Phi|_U = \sum_\alpha \phi^\alpha e_\alpha$ and Φ is defined by k functions $\phi^1, \ldots, \phi^k \in \mathcal{C}^\infty((J^1Y)_U)$, so that:

$$S \cap (J^1Y)_U = \{j^1_x s \in (J^1Y)_U / \phi^1(j^1_x s) = 0, \ldots, \phi^k(j^1_x s) = 0\}$$

where $(\partial\phi^\alpha/\partial y^i_\nu)$ has constant rank, k, along $S \cap (J^1Y)_U$.

In particular, if Φ is affine with respect to the corresponding affine structures of J^1Y and E on Y, we say that the constraint is affine. For affine constraints we have: $\phi^\alpha = a^\alpha + \sum_{i,\nu} b^{\nu\alpha}_i y^i_\nu$, where $a_\alpha, b^{\nu\alpha}_i \in \mathcal{C}^\infty(Y_U)$.

This morphism Φ can be seen as a section in $\Gamma(J^1Y, E_{J^1Y})$ (E_{J^1Y} the bundle over J^1Y induced from E by the projection π), in which case the constraint submanifold is simply the zero set of this section, i.e.: $S = \{j^1_x s / \Phi(j^1_x s) = 0\} \subset J^1Y$.

From this point the following characterization of $T_s(\Gamma_S(X, Y))$, the “tangent space” of $\Gamma_S(X, Y)$ at $s \in \Gamma_S(X, Y)$, is straightforward:

Proposition 3.4. *$D^{\mathrm{v}}_s \in T_s(\Gamma_S(X, Y))$ if and only if $(j^1 D^{\mathrm{v}}_s)\Phi = 0$, where the derivative of the section Φ is taken with respect to any linear connection on E_{J^1Y}.*

Proof. From the local expression $\Phi = \sum_\alpha \phi^\alpha e_\alpha$ follows, by covariant derivative with respect to $j^1 D^{\mathrm{v}}_s$:

$$(j^1 D^{\mathrm{v}}_s)\Phi = \sum_{\alpha=1}^{k} ((j^1 D^{\mathrm{v}}_s)\phi^\alpha) e_\alpha(j^1 s) + \phi^\alpha(j^1 s)((j^1 D^{\mathrm{v}}_s) e_\alpha)$$

And now, as s is admissible, $\phi^\alpha(j^1 s) = 0$, and whichever the sections $(j^1 D^{\mathrm{v}}_s)e_\alpha$ might be, we conclude that $(j^1 D^{\mathrm{v}}_s)\Phi = 0$ if and only if $(j^1 D^{\mathrm{v}}_s)\phi^\alpha = 0$, which is, precisely, the condition on $j^1 D^{\mathrm{v}}_s$ to be tangential to the submanifold S along $j^1 s$. □

To express now a second hypothesis that we shall impose on the constraint submanifold $S \subset J^1Y$, we shall first generalize the formalism developed in Section 2 taking as Lagrangian density the E_{J^1Y}-valued n-form $\Phi\omega$.

The two fundamental Propositions 2.3 and 2.4 can be generalized without essential modifications:

Proposition 3.5 (Constraint's momentum form). *There exists a unique $(VY^* \otimes E)_{J^1Y}$-valued $(n-1)$-form $\Omega_{\Phi\omega}$ on J^1Y such that $\Omega_{\Phi\omega} = i_F\omega$, where F is any $(VY^* \otimes E)_{J^1Y}$-valued vector field on J^1Y*, solution of the equation:

$$i_F \, \mathrm{d}\theta = \mathrm{d}\Phi$$

over the π-vertical vector fields of J^1Y, where the exterior derivative is taken with respect to the induced connections on J^1Y of two linear connections ∇ on VY and ∇_E on E and the bilinear products are the obvious ones. The $(n-1)$-form $\Omega_{\Phi\omega}$ does not depend on the choice of the connections ∇ and ∇_E.

Proof. Let e_α $(\alpha = 1, \ldots, k)$ be a local basis of the module of sections $\Gamma(Y, E)$ on a neighborhood coordinated by (x^ν, y^j, y^k_ν), where $\Phi = \sum_\alpha \phi^\alpha e_\alpha$, and let $\Gamma^k_{\nu i}$, $\bar{\Gamma}^k_{ji}$ be the coefficients of the connection ∇ on VY and $\gamma^\alpha_{\nu\beta}$, $\bar{\gamma}^\alpha_{j\beta}$ the coefficients of the connection ∇_E on E, i.e.:

$$\nabla_{\partial/\partial x^\nu}\left(\frac{\partial}{\partial y^i}\right) = \Gamma^k_{\nu i}\left(\frac{\partial}{\partial y^k}\right), \quad \nabla_{\partial/\partial y^j}\left(\frac{\partial}{\partial y^i}\right) = \bar{\Gamma}^k_{ji}\left(\frac{\partial}{\partial y^k}\right)$$

$$(\nabla_E)_{\partial/\partial x^\nu}(e_\beta) = \gamma^\alpha_{\nu\beta} e_\alpha, \qquad (\nabla_E)_{\partial/\partial y^j}(e_\beta) = \bar{\gamma}^\alpha_{j\beta} e_\alpha$$

where Einstein convention on repeated indexes is used here and from now on.

Using these expressions, imposing the conditions of our statement in local coordinates, the local expression of $\Omega_{\Phi\omega}$ is univocally determined:

$$\Omega_{\Phi\omega} = \frac{\partial \phi^\alpha}{\partial y^i_\nu} \omega_\nu \otimes \mathrm{d}y^i \otimes e_\alpha, \quad \omega_\nu = i_{(\partial/\partial x^\nu)}\omega \tag{3.1}$$

therefore, in virtue of the uniqueness, we conclude. □

As for the ordinary case, we may now define an E_{J^1Y}-valued Cartan n-form associated to the constraint by the rule:

$$\Theta_{\Phi\omega} = \theta \bar{\wedge} \Omega_{\Phi\omega} + \Phi\omega \tag{3.2}$$

with the following analogous property:

Proposition 3.6. *There exists an unique $(VY^* \otimes E)_{J^1Y}$-valued (j^1p)-horizontal n-form $\mathcal{F}_{\Phi\omega}$ on J^1Y such that*:

$$\mathrm{d}\Theta_{\Phi\omega} = \theta \bar{\wedge} (\mathcal{F}_{\Phi\omega} - \mathrm{d}\Omega_{\Phi\omega}) \tag{3.3}$$

where the exterior derivatives are taken with respect to the connections on J^1Y induced by a linear connection ∇ on *VY with vanishing vertical torsion and a linear connection ∇_E on E.*

Proof. In the local coordinate system from the proof of Proposition 3.5, Eq. (3.3) from the statement locally allows us to univocally determine $\mathcal{F}_{\Phi\omega}$ by the formula:

$$\mathcal{F}_{\Phi\omega} = \left[\frac{\partial\phi^\alpha}{\partial y^i} + \overline{\gamma}^\alpha_{i\beta}\phi^\beta - (\Gamma^j_{\nu i} + \overline{\Gamma}^j_{ki} y^k_\nu)\frac{\partial\phi^\alpha}{\partial y^j_\nu}\right]\omega\otimes \mathrm{d}y^i\otimes e_\alpha$$

therefore, in virtue of the uniqueness of this local expression, we conclude. □

We shall call the $(VY^*\otimes E)_{J^1Y}$-valued n-form $\mathbb{E}_{\Phi\omega} = \mathcal{F}_{\Phi\omega} - \mathrm{d}\Omega_{\Phi\omega}$ on J^1Y the Euler–Lagrange form associated to the constraint, which allows us to define the corresponding Euler–Lagrange operator $\mathcal{E}_{\Phi\omega} : s\in\Gamma(X,Y)\mapsto \mathcal{E}_{\Phi\omega}(s)\in\Gamma(X, s^*(VY^*\otimes E))$, by the rule:

$$\mathcal{E}_{\Phi\omega}(s)\otimes\omega = (j^1s)^*\mathbb{E}_{\Phi\omega}$$

Locally:

$$\mathcal{E}_{\Phi\omega}(s) = \left[\frac{\partial\phi^\alpha}{\partial y^i}\circ j^1s - \frac{\partial}{\partial x^\nu}\left(\frac{\partial\phi^\alpha}{\partial y^i_\nu}\circ j^1s\right)\right.$$
$$\left. + \left(\overline{\gamma}^\alpha_{i\beta}\phi^\beta - (\gamma^\alpha_{\nu\beta} + y^k_\nu\overline{\gamma}^\alpha_{k\beta})\frac{\partial\phi^\beta}{\partial y^i_\nu}\right)\circ j^1s\right]\mathrm{d}y^i\otimes e_\alpha \tag{3.4}$$

From this point, variation formulas (2.5), (2.6) and (2.7) from Section 2 can be generalized without any change to the E_{J^1Y}-valued n-form $\Phi\omega$. In particular, one gets:

$$L_{j^1D^{\mathrm{v}}_s}(\Phi\omega) = \mathcal{E}_{\Phi\omega}(s)(D^{\mathrm{v}}_s)\otimes\omega + \mathrm{d}(\Omega_{\Phi\omega}(s)(D^{\mathrm{v}}_s)),\qquad D^{\mathrm{v}}_s\in\Gamma(X, s^*VY) \tag{3.5}$$

which shall play an essential role in the determination of the tangent space $T_s(\Gamma_S(X,Y))$ at $s\in\Gamma_S(X,Y)$ of the space $\Gamma_S(X,Y)$. This is what we consider next.

The Euler–Lagrange operator $\mathcal{E}_{\Phi\omega}$ defines a section of the induced bundle $(VY^*\otimes E)_{J^2Y}$, while the momentum $(n-1)$-form $\Omega_{\Phi\omega}$ induces by pull-back a $(VY^*\otimes E)_{J^2Y}$-valued $(n-1)$-form on J^2Y. In these conditions, the second hypothesis we shall impose on the constraint submanifold $S=\{j^1_xs/\Phi(j^1_xs)=0\}\subset J^1Y$ is the following:

Hypothesis (HY2). On an open subset of J^1Y, dense in S, there exists a section $N\in\Gamma(J^2Y,(E^*\otimes VY)_{J^2Y})$ solution of the system of equations:

$$\Omega_{\Phi\omega}\circ N = 0,\qquad \mathcal{E}_{\Phi\omega}\circ N = \mathrm{I} \tag{3.6}$$

where $\circ$ stands for the bilinear product $(VY^*\otimes E)_{J^2Y}\times(E^*\otimes VY)_{J^2Y}\to(\mathrm{End}(E))_{J^2Y}$ defined by the composition of morphisms, and I is the identity endomorphism in $\Gamma(J^2Y,(\mathrm{End}(E))_{J^2Y})$.

In local coordinates, if $N = N^i_\alpha e^{*\alpha} \otimes (\partial/\partial y^i)$ (where $N^i_\alpha \in \mathcal{C}^\infty(J^2Y)$ and $e^{*\alpha}$ is the dual basis of a basis e_α of $\Gamma(J^2Y, E_{J^2Y})$), following (3.1) and (3.4), the system (3.6) may be expressed as:

$$\sum_{j=1}^{m} \frac{\partial \phi^\alpha}{\partial y^j_\nu} N^j_\beta = 0, \quad \sum_{j=1}^{m} \left[\frac{\partial \phi^\alpha}{\partial y^j} + \sum_\gamma \overline{\gamma}^\alpha_{j\gamma} \phi^\gamma - \sum_\nu \frac{\partial}{\partial x^\nu} \left(\frac{\partial \phi^\alpha}{\partial y^j_\nu} \right) \right] N^j_\beta = \delta^\alpha_\beta \tag{3.7}$$

(for $1 \le \alpha, \beta \le k$ and $1 \le \nu \le n$), which is a system of $k^2(n+1)$ linear equations with km unknowns that, in particular, for $k \le m/(n+1)$ and maximal rank for the matrix of the system, is compatible.

Remark.

1. Whereas Hypothesis (HY1) is the usual one in the setting of the problem of Lagrange, it is not so for Hypothesis (HY2), which is new and is justified, among other reasons, by the following Theorem 3.7, where the tangent space $T_s(\Gamma_S(X, Y))$ is characterized as the image of a linear differential operator on the space of sections $\Gamma(X, s^*VY)$. Though this condition introduces an additional restriction on the constraint submanifolds, it still leaves a wide margin for the application of this approach, as we shall see in Section 6 of this work.
2. As can be seen, the second group of equations (3.7) depends on the choice of the connection ∇_E on $q : E \to Y$, due to the term $\sum_\gamma \overline{\gamma}^\alpha_{j\gamma}\phi^\gamma$. This dependence disappears both when the equations are considered along the submanifold S (where $\phi^\alpha = 0$) and when E and ∇_E are obtained by pull-back to Y of a vector bundle on X and a connection on this bundle. In the different applications this is the usual case, so we will be in situation to eliminate this term.

The first important consequence of our hypothesis is, as we just mentioned, the following characterization of the tangent space $T_s(\Gamma_S(X, Y))$:

Theorem 3.7. *For any admissible section $s \in \Gamma_S(X, Y)$, the first order differential operator $P_s : \Gamma(X, s^*VY) \to \Gamma(X, s^*VY)$ defined by the rule*:

$$P_s(D^{\mathrm{v}}_s) = D^{\mathrm{v}}_s - N_s \circ (j^1 D^{\mathrm{v}}_s)\Phi, \quad D^{\mathrm{v}}_s \in \Gamma(X, s^*VY) \tag{3.8}$$

where $N_s \in \Gamma(X, s^*(E^* \otimes VY))$ is the value along j^2s of a solution N of the system (3.6), is a projector from $\Gamma(X, s^*VY)$ onto the real subspace $T_s(\Gamma_S(X, Y))$ of admissible infinitesimal variations at s, whose kernel is the $\mathcal{C}^\infty(X)$-submodule of $\Gamma(X, s^*VY)$ defined by the sections of the form $N_s(e)$, $e \in \Gamma(X, s^*E)$.

The first order differential operator $P^+_s : \Gamma(X, s^*VY^*) \to \Gamma(X, s^*VY^*)$ given by the rule:

$$P^+_s \mathcal{E}_s \otimes \omega = \mathcal{E}_s \otimes \omega + \lambda_{\mathcal{E}_s} \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda_{\mathcal{E}_s} \overline{\wedge} \Omega_{\Phi\omega}(s), \quad \mathcal{E}_s \in \Gamma(X, s^*VY^*) \tag{3.9}$$

where $\lambda_{\mathcal{E}_s} = -\mathcal{E}_s \circ N_s$, *satisfies the commutation rule*:

$$\mathcal{E}_s(P_s(D_s^{\mathrm{v}}))\omega = (P_s^{+}\mathcal{E}_s)(D_s^{\mathrm{v}})\omega + \mathrm{d}(\lambda_{\mathcal{E}_s} \circ \Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}})), \quad D_s^{\mathrm{v}} \in \Gamma(X, s^* VY) \quad (3.10)$$

Proof. By Proposition 3.4, P_s is the identity on $T_s(\Gamma_S(X, Y))$, therefore $T_s(\Gamma_S(X, Y)) \subseteq \mathrm{Im}P_s$.

Conversely, given an element $P_s(D_s^{\mathrm{v}})$ from $\mathrm{Im}P_s$, following formula (3.5) we get:

$$\begin{aligned} &L_{j^1(P_s(D_s^{\mathrm{v}}))}(\Phi\omega) \\ &\quad = \mathcal{E}_{\Phi\omega}(s)(P_s(D_s^{\mathrm{v}})) \otimes \omega + \mathrm{d}(\Omega_{\Phi\omega}(s)(P_s(D_s^{\mathrm{v}}))) \\ &\quad = \mathcal{E}_{\Phi\omega}(s)(D_s^{\mathrm{v}} - N_s \circ (j^1 D_s^{\mathrm{v}})\Phi) \otimes \omega + \mathrm{d}(\Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}} - N_s \circ (j^1 D_s^{\mathrm{v}})\Phi)) \\ &\quad = \mathcal{E}_{\Phi\omega}(s)(D_s^{\mathrm{v}}) \otimes \omega - \mathcal{E}_{\Phi\omega}(s)(N_s \circ (j^1 D_s^{\mathrm{v}})\Phi) \otimes \omega \\ &\qquad + \mathrm{d}[\Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}) - \Omega_{\Phi\omega}(s)(N_s \circ (j^1 D_s^{\mathrm{v}})\Phi)] \end{aligned}$$

now, due to the associativity of the bilinear products under consideration for the second and fourth terms of this expression, and due to the Eq. (3.6) satisfied by N, it holds:

$$\begin{aligned} &\mathcal{E}_{\Phi\omega}(s)(N_s \circ (j^1 D_s^{\mathrm{v}})\Phi) = (\mathcal{E}_{\Phi\omega}(s) \circ N_s)((j^1 D_s^{\mathrm{v}})\Phi) = (j^1 D_s^{\mathrm{v}})\Phi \\ &\Omega_{\Phi\omega}(s)(N_s \circ (j^1 D_s^{\mathrm{v}})\Phi) = (\Omega_{\Phi\omega}(s) \circ N_s)((j^1 D_s^{\mathrm{v}})\Phi) = 0 \end{aligned}$$

Therefore:

$$L_{j^1(P_s(D_s^{\mathrm{v}}))}(\Phi\omega) = \mathcal{E}_{\Phi\omega}(s)(D_s^{\mathrm{v}}) \otimes \omega - (j^1 D_s^{\mathrm{v}})\Phi \otimes \omega + \mathrm{d}(\Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}))$$

Taking now into account that $j^1 D_s^{\mathrm{v}}$ and $j^1(P_s(D_s^{\mathrm{v}}))$ are $j^1 p$-vertical, the previous result may be expressed as:

$$(j^1(P_s(D_s^{\mathrm{v}}))\Phi) \otimes \omega = \mathcal{E}_{\Phi\omega}(s)(D_s^{\mathrm{v}}) \otimes \omega - L_{j^1 D_s^{\mathrm{v}}}(\Phi\omega) + \mathrm{d}(\Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}))$$

and finally, applying formula (3.5) again, yields $j^1(P_s(D_s^{\mathrm{v}}))\Phi = 0$, which by Proposition 3.4 means that $P_s(D_s^{\mathrm{v}}) \in T_s(\Gamma_S(X, Y))$.

Furthermore, if D_s^{v} is in the kernel of P_s, then $D_s^{\mathrm{v}} = N_s \circ (j^1 D_s^{\mathrm{v}})\Phi$ and thus has the form $N_s(e)$, $e \in \Gamma(X, s^* E)$. Conversely, if we have a vector field of the form $D_s^{\mathrm{v}} = N_s(e)$, then:

$$\begin{aligned} P_s(N_s(e)) \otimes \omega &= N_s(e) \otimes \omega - N_s \circ (j^1(N_s(e))\Phi) \otimes \omega \\ &= N_s(e) \otimes \omega - N_s \circ L_{j^1(N_s(e))}(\Phi\omega) \\ &= N_s(e) \otimes \omega - N_s \circ [\mathcal{E}_{\Phi\omega}(s)(N_s(e)) \otimes \omega + \mathrm{d}(\Omega_{\Phi\omega}(s)(N_s(e)))] \end{aligned}$$

If we apply again (3.6) we get that $\Omega_{\Phi\omega}(s)(N_s(e)) = 0$ and $\mathcal{E}_{\Phi\omega}(s)(N_s(e)) = e$, therefore:

$$P_s(N_s(e)) = 0$$

Finally, to prove the last part of the statement, if we follow formula (3.5) and Definition (3.8) of P_s:

$$\begin{aligned}\mathcal{E}_s(P_s(D_s^{\mathrm{v}}))\omega &= [\mathcal{E}_s(D_s^{\mathrm{v}}) - \mathcal{E}_s(N_s \circ (j^1 D_s^{\mathrm{v}})\Phi)]\omega \\ &= \mathcal{E}_s(D_s^{\mathrm{v}})\omega - (\mathcal{E}_s \circ N_s) \circ L_{j^1 D_s^{\mathrm{v}}}(\Phi\omega) \\ &= \mathcal{E}_s(D_s^{\mathrm{v}})\omega + \lambda_{\mathcal{E}_s}[\mathcal{E}_{\Phi\omega}(s)(D_s^{\mathrm{v}}) \otimes \omega + \mathrm{d}(\Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}))] \\ &= \mathcal{E}_s(D_s^{\mathrm{v}})\omega + \lambda_{\mathcal{E}_s} \circ \mathcal{E}_{\Phi\omega}(s)(D_s^{\mathrm{v}}) \otimes \omega - \mathrm{d}\lambda_{\mathcal{E}_s} \bar{\wedge} \Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}) \\ &\quad + \mathrm{d}(\lambda_{\mathcal{E}_s} \circ \Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}})) = (P_s^+ \mathcal{E}_s)(D_s^{\mathrm{v}})\omega + \mathrm{d}(\lambda_{\mathcal{E}_s} \circ \Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}))\end{aligned}$$

thus proving the theorem. □

Remark.

1. An important fact to be emphasized about formula:

$$T_s(\Gamma_S(X, Y)) = P_s(\Gamma(X, s^* VY))$$

   is that, from the knowledge of a solution of the system of linear equations (3.6) (easy to compute if compatible), one can obtain the general solution of the system of partial differential equations $(j^1 D_s^{\mathrm{v}})\Phi = 0$ for $D_s^{\mathrm{v}} \in \Gamma(X, s^* VY)$ (which defines $T_s(\Gamma_S(X, Y))$) as the image of $\Gamma(X, s^* VY)$ by a certain differential operator. Regardless of its generality (this holds for arbitrary $\dim X$), this construction differs notably from the usual parameterizations (for $\dim X = 1$) of $T_s(\Gamma_S(X, Y))$, which are based on a suitable integration of $(j^1 D_s^{\mathrm{v}})\Phi = 0$ with initial conditions.
2. Taking compact-supported sections, there easily follows:

$$T_s^{\mathrm{c}}(\Gamma_S(X, Y)) = P_s(\Gamma^{\mathrm{c}}(X, s^* VY))$$

   and integrating (3.10) over X:

$$\int_X \mathcal{E}_s(P_s(D_s^{\mathrm{v}}))\omega = \int_X (P_s^+ \mathcal{E}_s)(D_s^{\mathrm{v}})\omega$$

   for any $D_s^{\mathrm{v}} \in \Gamma^{\mathrm{c}}(X, s^* VY)$, $\mathcal{E}_s \in \Gamma(X, s^* VY^*)$. Thus, P_s^+ coincides with the formal adjoint of the operator P_s.

Coming back to the general formalism, we now obtain the following fundamental result:

Theorem 3.8 (Constrained first variation formula). *For any admissible section* $s \in \Gamma_S(X, Y)$ *and admissible variation* $D_s^{\mathrm{v}} \in T_s(\Gamma_S(X, Y))$, *it holds*:

$$(\delta_s \mathbb{L})(D_s^{\mathrm{v}}) = \int_X (P_s^+ \mathcal{E}_{\mathcal{L}\omega}(s))(\bar{D}_s^{\mathrm{v}})\omega + \mathrm{d}[\Omega_{\mathcal{L}\omega}(s)(D_s^{\mathrm{v}}) + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \circ \Omega_{\Phi\omega}(s)(\bar{D}_s^{\mathrm{v}})]$$

where $\bar{D}_s^{\mathrm{v}} \in \Gamma(X, s^*VY)$ *is any section such that* $P_s(\bar{D}_s^{\mathrm{v}}) = D_s^{\mathrm{v}}$.

In particular:

$$(\delta_s \mathbb{L})(D_s^{\mathrm{v}}) = \int_X (P_s^+ \mathcal{E}_{\mathcal{L}\omega}(s))(\bar{D}_s^{\mathrm{v}})\omega, \quad D_s^{\mathrm{v}} \in T_s^{\mathrm{c}}(\Gamma_S(X, Y))$$

where $\bar{D}_s^{\mathrm{v}} \in \Gamma^{\mathrm{c}}(X, s^*VY)$ *is any section with compact support satisfying* $D_s^{\mathrm{v}} = P_s(\bar{D}_s^{\mathrm{v}})$.

Proof. The first formula is obtained by application of (2.7) to $D_s^{\mathrm{v}} = P_s(\bar{D}_s^{\mathrm{v}})$, taking into account (3.10), and the second one is then obtained taking compact-supported fields. □

The arbitrariness of $\bar{D}_s^{\mathrm{v}} \in \Gamma^{\mathrm{c}}(X, s^*VY)$ in the previous formula finally yields the following characterization of critical sections:

Corollary 3.9 (Euler–Lagrange equations). *A section* $s \in \Gamma_S(X, Y)$ *is critical for the constrained variational problem with Lagrangian density* $\mathcal{L}\omega$ *on* J^1Y *and constraint submanifold* $S = \{j_x^1 s / \Phi(j_x^1 s) = 0\} \subset J^1Y$ *satisfying* Hypotheses (HY1) and (HY2) *if and only if*:

$$P_s^+(\mathcal{E}_{\mathcal{L}\omega}(s)) \otimes \omega = \mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \bar{\wedge} \Omega_{\Phi\omega}(s) = 0 \tag{3.11}$$

where $\mathcal{E}_{\mathcal{L}\omega}$ *is the Euler–Lagrange operator of* $\mathcal{L}\omega$ *as an unconstrained variational problem, and* $\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} = -\mathcal{E}_{\mathcal{L}\omega}(s) \circ N_s$.

A remarkable fact about this characterization is that we have an explicit third order differential operator: $s \in \Gamma(X, Y) \mapsto \tilde{\mathcal{E}}(s) = P_s^+(\mathcal{E}_{\mathcal{L}\omega}(s)) \in \Gamma(X, s^*VY^*)$ (Euler–Lagrange operator) that, together with the constraints, provide us the set of critical sections of the constrained variational problem under consideration:

$$\Phi(s) = 0, \quad \tilde{\mathcal{E}}(s) = 0$$

Moreover, there exists a $VY^*_{J^2Y}$-valued n-form on J^2Y which we shall call Euler–Lagrange form:

$$\tilde{\mathbb{E}} = \mathbb{E}_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}} \circ \mathbb{E}_{\Phi\omega} - \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}} \bar{\wedge} \Omega_{\Phi\omega}, \quad \lambda_{\mathcal{E}_{\mathcal{L}\omega}} = -\mathcal{E}_{\mathcal{L}\omega} \circ N \tag{3.12}$$

such that, for any section $s \in \Gamma_S(X, Y)$, it holds:

$$\tilde{\mathcal{E}}(s) \otimes \omega = (j^2 s)^* \tilde{\mathbb{E}} \tag{3.13}$$

We must emphasize that such a characterization has been possible due to the consideration of the “universal multiplier”:

$$\lambda_{\mathcal{E}_{\mathcal{L}\omega}} = -\mathcal{E}_{\mathcal{L}\omega} \circ N \in \Gamma(J^2Y, E^*_{J^2Y}) \tag{3.14}$$

obtained from a solution N of the system of linear equations (3.6) and from the Euler–Lagrange operator $\mathcal{E}_{\mathcal{L}\omega}$ associated to $\mathcal{L}\omega$ as an unconstrained variational problem.

Remark.

1. Eq. (3.11) proves that, for any critical section $s \in \Gamma_S(X, Y)$, there exists a section $\lambda(s) = -\mathcal{E}_{\mathcal{L}\omega}(s) \circ N_s \in \Gamma(X, s^*E^*)$ such that:

$$\mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda(s) \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda(s) \bar{\wedge} \Omega_{\Phi\omega}(s) = 0 \tag{3.15}$$

This section $\lambda(s)$ is unique, indeed if $\lambda'(s)$ also satisfies (3.15), the difference $\eta(s) = \lambda'(s) - \lambda(s)$ would satisfy the equation:

$$\eta(s) \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\eta(s) \bar{\wedge} \Omega_{\Phi\omega}(s) = 0 \tag{3.16}$$

and, composing with N_s and taking into account Hypothesis (HY2) (Eq. (3.6)), we get $0 = \eta(s) \circ \mathcal{E}_{\Phi\omega}(s) \circ N_s \otimes \omega - \mathrm{d}\eta(s) \bar{\wedge} \Omega_{\Phi\omega}(s) \circ N_s = \eta(s) \otimes \omega$, that is, $\eta(s) = 0$ and $\lambda'(s) = \lambda(s)$.

Regarding this, we easily see that the operator $\varphi_s : \Gamma(X, s^*E^*) \to \Gamma(X, s^*VY^*)$ defined by the rule:

$$\varphi_s \eta \otimes \omega = \eta \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\eta \bar{\wedge} \Omega_{\Phi\omega}(s), \quad \eta \in \Gamma(X, s^*E) \tag{3.17}$$

is the adjoint of the operator $D_s \in \Gamma(X, s^*VY) \mapsto (j^1 D_s)\Phi \in \Gamma(X, s^*E)$, whose kernel is, precisely, $T_s(\Gamma_S(X, Y))$ (Proposition 3.4). The injectivity of φ_s, which we have just proved, implies for problems in one independent variable ($X = \mathbb{R}$) that the section s is regular (see [23] or, more recently, [25, Theorem 6]). This suggests the consideration of this injectivity condition as a suitable instrument to explore the conditions for $s \in \Gamma_S(X, Y)$ to be a regular solution of $\mathrm{Im}\, j^1 s \subset S$ in the general case.
2. Taking the corresponding variation formulas (Section 2 from [9]), Theorem 3.8, Corollary 3.9 and all the subsequent considerations also hold for r-order Lagrangian densities $\mathcal{L}\omega$, by only substituting $\mathcal{E}_{\mathcal{L}\omega}$, $\Omega_{\mathcal{L}\omega}$ and $\mathbb{E}_{\mathcal{L}\omega}$ by the corresponding objects of higher order variational calculus.

4. Cartan form. Cartan equation. Noether Theorem

The multiplier (3.14) allows us to establish a Cartan formalism for this class of constrained variational problems, proceeding as follows:

Definition 4.1. We shall call Cartan form of the constrained variational problem the ordinary n-form $\widetilde{\Theta}$ on J^2Y given by:

$$\widetilde{\Theta} = \Theta_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}} \circ \Theta_{\Phi\omega} \tag{4.1}$$

where $\Theta_{\mathcal{L}\omega}$ and $\Theta_{\Phi\omega}$ are respectively the pull-back to J^2Y of the Cartan forms (2.3) and (3.2), $\lambda_{\mathcal{E}_{\mathcal{L}\omega}} \in \Gamma(J^2Y, E^*_{J^2Y})$ is the section defined by formula (3.14) and $\circ$ is the duality bilinear product between the vector bundles E_{J^2Y} and $E^*_{J^2Y}$.

Analogous to formula (2.4) of unconstrained variational calculus, we obtain the following:

Proposition 4.2.

$$\mathrm{d}\widetilde{\Theta} = \theta \overline{\wedge} \widetilde{\mathbb{E}} + \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}} \overline{\wedge} \Phi\omega \tag{4.2}$$

where $\widetilde{\mathbb{E}}$ *is the Euler–Lagrange form* (3.12) *of the constrained variational problem.*

Proof. It is enough to compute $\mathrm{d}\widetilde{\Theta}$ using the differential calculus from Sections 2 and 3 and to apply formulas (2.4), (3.3) and (3.12). □

Definition 4.1 of Cartan form is justified by the following fundamental result:

Theorem 4.3 (Cartan equation). *An admissible section* $s \in \Gamma_S(X, Y)$ *is critical for the constrained variational problem if and only if*:

$$(j^2 s)^*(i_D \mathrm{d}\widetilde{\Theta}) = 0, \quad \forall D \in \mathfrak{X}(J^2 Y) \tag{4.3}$$

equivalently: *if and only* (4.3) *holds for any vector field* $D \in \mathfrak{X}(S^{(2)})$, *where* $S^{(2)} \subset J^2 Y$ *is the inverse image of the constraint submanifold* $S \subset J^1 Y$ *by the canonical projection* $j^1\pi : J^2 Y \to J^1 Y$.

Proof. Given $s \in \Gamma_S(X, Y)$ and $D \in \mathfrak{X}(J^2 Y)$, by Proposition 4.2, taking into account that $(j^1 s)^*\theta = 0$, $(j^1 s)^*\Phi = 0$, and formula (3.13), we get:

$$(j^2 s)^*(i_D \mathrm{d}\widetilde{\Theta}) = (j^2 s)^*(\theta(D) \circ \widetilde{\mathbb{E}}) = \widetilde{\mathcal{E}}(s)(D_s^{\mathrm{v}})\omega$$

where $D_s^{\mathrm{v}} \in \Gamma(X, s^* VY)$ is the p-vertical component along s of the projection on Y of the vector field $D_{j^2 s}$ along $j^2 s$ defined by D.

Taking now into account that the mapping $D \in \mathfrak{X}(J^2 Y) \mapsto D_s^{\mathrm{v}} \in \Gamma(X, s^* VY)$ is surjective, then s is critical (i.e. $\widetilde{\mathcal{E}}(s) = 0$) if and only if $(j^2 s)^*(i_D \mathrm{d}\widetilde{\Theta}) = 0$ for any vector field $D \in \mathfrak{X}(J^2 Y)$.

Only remains to prove that from the (weaker) condition $(j^2 s)^*(i_D\, \mathrm{d}\widetilde{\Theta}) = 0$ for any vector field $D \in \mathfrak{X}(S^{(2)})$ also follows that s is critical. Indeed, under this hypothesis and taking into account that the mapping $D \in \mathfrak{X}(S^{(2)}) \mapsto D_s^{\mathrm{v}} \in T_s(\Gamma_S(X, Y))$ is surjective, the previous formula $(j^2 s)^*(i_D\, \mathrm{d}\widetilde{\Theta}) = \widetilde{\mathcal{E}}(s)(D_s^{\mathrm{v}})\omega$ together with Theorem 3.7 allow us to state that $\widetilde{\mathcal{E}}(s)(P_s(D_s^{\mathrm{v}})) = 0$ for any $D_s^{\mathrm{v}} \in \Gamma(X, s^* VY)$. From this, due to commutation formula (3.10) and being P_s^+ a projector (so $P_s^+ \widetilde{\mathcal{E}}(s) = P_s^+ P_s^+ \mathcal{E}_{\mathcal{L}\omega}(s) = P_s^+ \mathcal{E}_{\mathcal{L}\omega}(s) = \widetilde{\mathcal{E}}(s)$), we obtain:

$$0 = \widetilde{\mathcal{E}}(s)(P_s(D_s^{\mathrm{v}}))\omega = \widetilde{\mathcal{E}}(s)(D_s^{\mathrm{v}})\omega + \mathrm{d}(\lambda_{\widetilde{\mathcal{E}}(s)} \circ \Omega_{\Phi\omega}(s)(D_s^{\mathrm{v}}))$$

Taking now, in particular, sections D_s^{v} with compact support, and integrating along X, we obtain:

$$\int_X \widetilde{\mathcal{E}}(s)(D_s^{\mathrm{v}})\omega = 0, \quad D_s^{\mathrm{v}} \in \Gamma^{\mathrm{c}}(X, s^* VY)$$

and hence $\widetilde{\mathcal{E}}(s) = 0$, i.e., s is critical. □

In this framework, all the typical questions from unconstrained variational calculus can be developed in a similar way. In particular, the notion of infinitesimal symmetry and Noether Theorem can be established as follows:

Definition 4.4. An infinitesimal symmetry of the constrained variational problem is a vector field $D \in \mathfrak{X}(Y)$ such that:

$$L_{j^1 D}(\mathcal{L}\omega) = 0, \quad j^1 D \text{ is tangential to } S$$

Theorem 4.5 (Noether). *If $s \in \Gamma_S(X, Y)$ is a critical section and D is an infinitesimal symmetry of the constrained variational problem, then*:

$$\mathrm{d}[(j^2 s)^* i_{j^2 D} \widetilde{\Theta}] = 0$$

Proof. From $L_{j^1 D}(\mathcal{L}\omega) = 0$ follows:

$$(j^1 s)^* L_{j^1 D} \Theta_{\mathcal{L}\omega} = (j^1 s)^* L_{j^1 D}(\theta \bar{\wedge} \Omega_{\mathcal{L}\omega} + \mathcal{L}\omega) = (j^1 s)^* (\theta \bar{\wedge} \eta + L_{j^1 D} \mathcal{L}\omega) = 0$$

On the other hand, as $j^1 D$ is tangential to S, $(j^1 D)\Phi = 0$ holds along S, which, together with the annihilation of Φ along S, yields:

$$\begin{aligned}
&(j^1 s)^* \Theta_{\Phi\omega} = (j^1 s)^* (\theta \bar{\wedge} \Omega_{\Phi\omega} + \Phi\omega) = 0 \\
&(j^1 s)^* L_{j^1 D} \Theta_{\Phi\omega} = (j^1 s)^* L_{j^1 D}(\theta \bar{\wedge} \Omega_{\phi\omega} + \Phi\omega) \\
&\quad = (j^1 s)^* (\theta \bar{\wedge} \eta' + (j^1 D)\Phi\omega + \Phi L_{j^1 D}\omega) = 0
\end{aligned}$$

From the previous three equations follows:

$$\begin{aligned}
(j^2 s)^* L_{j^2 D} \widetilde{\Theta} &= (j^2 s)^* L_{j^2 D}(\Theta_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}} \circ \Theta_{\Phi\omega}) \\
&= (j^1 s)^* L_{j^1 D} \Theta_{\mathcal{L}\omega} + (j^2 s)^* ((j^2 D)\lambda_{\mathcal{E}_{\mathcal{L}\omega}} \circ \Theta_{\Phi\omega}) \\
&\quad + (j^2 s)^* (\lambda_{\mathcal{E}_{\mathcal{L}\omega}} \circ L_{j^1 D} \Theta_{\Phi\omega}) = 0
\end{aligned}$$

Now, as s is critical, Cartan equation yields $(j^2 s)^*[i_{j^2 D}\, \mathrm{d}\widetilde{\Theta}] = 0$, therefore:

$$\mathrm{d}[(j^2 s)^* i_{j^2 D} \widetilde{\Theta}] = (j^2 s)^* \mathrm{d} i_{j^2 D} \widetilde{\Theta} = (j^2 s)^* L_{j^2 D} \widetilde{\Theta} = 0$$

as we wanted. □

If we denote by $\mathrm{Sym}_S(\mathcal{L}\omega)$ the real Lie algebra of infinitesimal symmetries of the constrained variational problem, we can now introduce the concept of multi-momentum map for this kind of problems as follows:

Definition 4.6. The multi-momentum map associated to the constrained variational problem $(\mathcal{L}\omega, S \subset J^1 Y)$ is the mapping $\tilde{\mu}_{\mathcal{L}\omega} : \Gamma_S(X, Y) \to \mathrm{Sym}_S(\mathcal{L}\omega)^* \otimes \Gamma(X, \Lambda^{n-1} T^* X)$ defined by the rule:

$$\tilde{\mu}_{\mathcal{L}\omega}(s)(D) = (j^2 s)^* i_{j^2 D}\widetilde{\Theta} = (j^1 s)^* i_{j^1 D}\Theta_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \circ (j^1 s)^* i_{j^1 D}\Theta_{\Phi\omega}$$

for any $D \in \mathrm{Sym}_S(\mathcal{L}\omega)$.

5. Projectability of the Cartan form. Relation with the Lagrange multiplier rule. Regularity

The multiplier $\lambda_{\mathcal{E}_{\mathcal{L}\omega}} = -\mathcal{E}_{\mathcal{L}\omega} \circ N \in \Gamma(J^2Y, E^*_{J^2Y})$ allows us to define a bundle morphism:

$$\begin{array}{ccc} J^2Y & \xrightarrow{\varphi} & J^1Y \times_Y E^* \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \tag{5.1}$$

by the rule:

$$\varphi(j_x^2 s) = (j_x^1 s, \lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j_x^2 s))$$

Via this morphism the Cartan form $\widetilde{\Theta}$ of the constrained variational problem is projected to the n-form on $J^1Y \times_Y E^*$:

$$\widehat{\Theta} = \Theta_{\mathcal{L}\omega} + \lambda \circ \Theta_{\Phi\omega} \tag{5.2}$$

where $\lambda \in \Gamma(J^1Y \times_Y E^*, E^*_{J^1Y\times_Y E^*})$ is the tautological section $\lambda(j_x^1 s, e^*_{s(x)}) = e^*_{s(x)}$ and $\circ$ is the bilinear duality product. That is, it holds:

$$\widetilde{\Theta} = \varphi^* \widehat{\Theta} \tag{5.3}$$

In these conditions, critical sections of the constrained variational problem admit the following new characterization:

Theorem 5.1. *A section $s \in \Gamma_S(X, Y)$ is critical for the constrained variational problem if and only if the section $\widehat{s} = \varphi \circ j^2 s = (j^1 s, \lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j^2 s)) \in \Gamma(X, S \times_Y E^*)$ satisfies Cartan equation*:

$$\widehat{s}^*(i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta}) = 0, \quad \forall \widehat{D} \in \mathfrak{X}(J^1Y \times_Y E^*) \tag{5.4}$$

or equivalently, if and only if (5.4) holds for any vector field $\widehat{D} \in \mathfrak{X}(S \times_Y E^*)$.

Proof. Let $s \in \Gamma(X, Y)$ be an admissible critical section for the constrained variational problem, and $\widehat{s} = \varphi \circ j^2 s = (j^1 s, \lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j^2 s)) \in \Gamma(X, S \times_Y E^*)$. As the canonical projection $J^2Y \to J^1Y$ is regular and coincides with the composition of φ with the canonical projection $J^1Y \times_Y E^* \to J^1Y$, we get the decomposition of the space $\mathfrak{X}_{\widehat{s}}(J^1Y \times_Y E^*)$ of vector fields on $J^1Y \times_Y E^*$ along $\widehat{s}$:

$$\mathfrak{X}_{\widehat{s}}(J^1Y \times_Y E^*) = \varphi_*(\mathfrak{X}_{j^2s}(J^2Y)) + \mathfrak{X}_{\widehat{s}}(E^*)$$

where $\mathfrak{X}_{\widehat{s}}(E^*)$ is the space of vector fields along $\widehat{s}$ whose projection to J^1Y vanishes.

Following (5.3) and Cartan equation (4.3), for any vector field $D_{j^2s} \in \mathfrak{X}_{j^2s}(J^2Y)$, it holds:

$$(\widehat{s})^* i_{\varphi_* D_{j^2s}} \, \mathrm{d}\widehat{\Theta} = (j^2s)^* \circ \varphi^*(i_{\varphi_* D_{j^2s}} \, \mathrm{d}\widehat{\Theta}) = (j^2s)^*(i_{D_{j^2s}} \, \mathrm{d}\widetilde{\Theta}) = 0$$

On the other hand, for any vector field $D_{E^*} \in \mathfrak{X}_{\widehat{s}}(E^*)$, as s is admissible, following (5.2) we have:

$$\begin{aligned}\widehat{s}^*(i_{D_{E^*}} \, \mathrm{d}\widehat{\Theta}) &= \widehat{s}^* i_{D_{E^*}}(\mathrm{d}\Theta_{\mathcal{L}\omega} + \mathrm{d}\lambda \overline{\wedge} \Theta_{\Phi\omega} + \lambda \circ \mathrm{d}\Theta_{\Phi\omega}) \\ &= \widehat{s}^*(i_{D_{E^*}} \, \mathrm{d}\lambda) \circ \Phi(j^1s) \otimes \omega = 0\end{aligned}$$

Hence, for any vector field $\widehat{D} = \varphi_* D_{j^2s} + D_{E^*}$ along $\widehat{s}$, it holds:

$$\widehat{s}^*(i_{\widehat{D}} \, \mathrm{d}\widehat{\Theta}) = \widehat{s}^* i_{\varphi_* D_{j^2s}} \, \mathrm{d}\widehat{\Theta} + \widehat{s}^* i_{D_{E^*}} \, \mathrm{d}\widehat{\Theta} = 0$$

which proves our statement in one direction for both cases.

Conversely, let $s \in \Gamma_S(X, Y)$ satisfy (5.4). Applying now Propositions 2.4 and 3.6, $\mathrm{d}\widehat{\Theta}$ can also be computed in the following way:

$$\mathrm{d}\widehat{\Theta} = \theta \overline{\wedge}(\mathbb{E}_{\mathcal{L}\omega} + \lambda \circ \mathbb{E}_{\Phi\omega} - \mathrm{d}\lambda \overline{\wedge} \Omega_{\Phi\omega}) + \mathrm{d}\lambda \overline{\wedge} \Phi\omega \tag{5.5}$$

so now, as s is admissible, we have $\Phi(j^1s) = 0$ and taking in Eq. (5.4) an arbitrary vector field $\widehat{D} \in \mathfrak{X}(J^1Y \times_Y E^*)$:

$$\begin{aligned}0 &= \widehat{s}^*(i_{\widehat{D}} \, \mathrm{d}\widehat{\Theta}) = \widehat{s}^*(\theta(\widehat{D}) \circ (\mathbb{E}_{\mathcal{L}\omega} + \lambda \mathbb{E}_{\Phi\omega} - \mathrm{d}\lambda \overline{\wedge} \Omega_{\Phi\omega})) \\ &= (\mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \overline{\wedge} \Omega_{\Phi\omega}(s))(D_s^{\mathrm{v}}) = \widetilde{\mathcal{E}}(s)(D_s^{\mathrm{v}}) \otimes \omega\end{aligned}$$

where $D_s^{\mathrm{v}} \in \Gamma(X, s^*VY)$ is the vertical component along s of the projection D_s on Y of the vector field $\widehat{D}_{\widehat{s}}$ along $\widehat{s}$.

From this point, due to the arbitrariness of D_s^{v}, $\widetilde{\mathcal{E}}(s) = 0$ and therefore, following Corollary 3.9, the section $s \in \Gamma_S(X, Y)$ is critical.

There only remains to prove that the (weaker) condition $\widehat{s}^*(i_{\widehat{D}} \, \mathrm{d}\widehat{\Theta}) = 0$ for any $\widehat{D} \in \mathfrak{X}(S \times_Y E^*)$ also implies that s is critical. In fact, from the previous computations and Theorem 3.7, under this condition we obtain $\widetilde{\mathcal{E}}(s)(P_s(D_s^{\mathrm{v}})) = 0$ for any $D_s^{\mathrm{v}} \in \Gamma(X, s^*VY)$ and proceeding in the same way as for the second part of the proof of Theorem 4.3 we conclude that $\widetilde{\mathcal{E}}(s) = 0$, that is, $s \in \Gamma_S(X, Y)$ is critical. □

Remark. It is important to note that it holds a corresponding projection of Noether Theorem. Indeed, for any vector field $D \in \mathfrak{X}(Y)$, it holds:

$$i_{j^2D}\widetilde{\Theta} = i_{j^2D}(\Theta_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}} \circ \Theta_{\Phi\omega}) = i_{j^1D}\Theta_{\mathcal{L}\omega} + (\varphi^*\lambda) \circ i_{j^1D}\Theta_{\Phi\omega}$$
$$= \varphi^*(i_{j^1D}\Theta_{\mathcal{L}\omega} + \lambda \circ i_{j^1D}\Theta_{\Phi\omega}) = \varphi^* i_{j^1D}\widehat{\Theta}$$

and therefore for any section $\widehat{s} = \varphi \circ j^2 s$, $s \in \Gamma(X, Y)$:

$$\widehat{s}^* i_{j^1D}\widehat{\Theta} = (j^2 s)^* \circ \varphi^* i_{j^1D}\widehat{\Theta} = (j^2 s)^* i_{j^2D}\widetilde{\Theta}$$

In particular, if $D \in \mathrm{Sym}_S(\mathcal{L}\omega)$ and $s \in \Gamma_S(X, Y)$ is critical for the constrained variational problem, following Theorem 4.5:

$$0 = \mathrm{d}((j^2 s)^* i_{j^2D}\widetilde{\Theta}) = \mathrm{d}(\widehat{s}^* i_{j^1D}\widehat{\Theta}) \tag{5.6}$$

which is the projected Noether Theorem we wanted to prove.

The relation of these results with the "Lagrange multiplier rule" is straightforward:

Let us consider the unconstrained variational problem with Lagrangian density $(\mathcal{L} + \lambda \circ \Phi)\omega$ on $J^1(Y \times_Y E^*)$, where we write $Y \times_Y E^*$ instead of E^* to keep in mind the two components involved, and where we still denote by λ the pull-back of the tautological section by the bundle morphism:

$$\begin{array}{ccc} J^1(Y \times_Y E^*) & \xrightarrow{\ \psi\ } & J^1 Y \times_Y E^* \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \tag{5.7}$$

Simple computations allow us to prove that the Euler–Lagrange equations satisfied by critical sections $(s, \lambda) \in \Gamma(X, Y \times_Y E^*)$ of this variational problem are:

$$\Phi(j^1 s) = 0, \quad \mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda \,\bar{\wedge}\, \Omega_{\Phi\omega}(s) = 0 \tag{5.8}$$

Thus, if $s \in \Gamma_S(X, Y)$ is a critical section of the constrained variational problem, i.e. $\widetilde{\mathcal{E}}(s) = 0$, the section $(s, \lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j^2 s)) \in \Gamma(X, Y \times_Y E^*)$ satisfies Eq. (5.8), and conversely, if $(s, \lambda) \in \Gamma(X, Y \times_Y E^*)$ satisfies these equations, by composition of the second equation in (5.8) with N_s and taking into account (3.6) we obtain $\lambda = \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)}$, therefore $s \in \Gamma_S(X, Y)$ and $\widetilde{\mathcal{E}}(s) = 0$, that is, s is a critical section for the constrained variational problem.

The mapping $\Pi : (s, \lambda) \in \Gamma(X, Y \times_Y E^*) \mapsto s \in \Gamma(X, Y)$ defines then a canonical bijection:

$$\Pi : \Gamma_{\mathrm{crit}}(X, Y \times_Y E^*) \xrightarrow{\ \sim\ } \Gamma_{\mathrm{crit}}(X, Y) \tag{5.9}$$

between the set of critical sections of the unconstrained variational problem $(\mathcal{L} + \lambda \circ \Phi)\omega$ and the set of critical sections of the constrained variational problem $(\mathcal{L}\omega, S \subset J^1 Y)$, which constitutes the expression in this formalism of the Lagrange multiplier rule.

Remark. The standard way to prove that any critical section $(s, \lambda) \in \Gamma(X, Y \times_Y E^*)$ of the unconstrained variational problem $(\mathcal{L} + \lambda \circ \Phi)\omega$ is also critical for the constrained variational problem $(\mathcal{L}\omega, S \subset J^1 Y)$ is, as is well known, the following: from the first equation in (5.8), $s \in \Gamma_S(X, Y)$, and taking arbitrary vector fields $D \in \mathfrak{X}_{(s,\lambda)}(Y \times_Y E^*)$ along (s, λ) whose projection to Y is $D^{\mathrm{v}}_s \in T^{\mathrm{c}}_s(\Gamma_S(X, Y))$, i.e. such that $(j^1 D^{\mathrm{v}}_s)\Phi = 0$, one gets:

$$0 = \int_{j^1(s,\lambda)} L_{j^1 D}(\mathcal{L} + \lambda \circ \Phi)\omega = \int_{j^1 s} L_{j^1 D^{\mathrm{v}}_s} \mathcal{L}\omega$$

therefore $s \in \Gamma_S(X, Y)$ is critical for the constrained variational problem.

However, this only proves that the mapping (5.9) is well defined, but it need not be injective nor surjective. It is precisely to obtain the latter that one needs to impose additional conditions. In particular, our Hypothesis (HY2) allows us to prove the fundamental bijection (5.9) and, which is more attractive, in a purely differential-geometric way.

Regarding the relation between the Cartan forms involved in both formalisms, one gets the following result:

Proposition 5.2. *The Cartan form $\Theta_{(\mathcal{L}+\lambda\circ\Phi)\omega}$ of the Lagrangian density $(\mathcal{L} + \lambda \circ \Phi)\omega$ projects via the bundle morphism ψ to the form $\widehat{\Theta}$. That is*:

$$\Theta_{(\mathcal{L}+\lambda\circ\Phi)\omega} = \psi^* \widehat{\Theta}$$

Proof. Taking the differential of $\mathcal{L} + \lambda \circ \Phi$ and taking into account that $\mathrm{d}\lambda$ vanishes on vector fields of $J^1(Y \times_Y E^*)$ which are vertical over $Y \times_Y E^*$, one gets by Proposition 2.3:

$$\Omega_{(\mathcal{L}+\lambda\circ\Phi)\omega} = \Omega_{\mathcal{L}\omega} + \lambda \circ \Omega_{\Phi\omega}$$

where we still denote by $\Omega_{\mathcal{L}\omega}$ and $\Omega_{\Phi\omega}$ the pull-backs to $J^1(Y \times_Y E^*)$ of the corresponding momentum forms on $J^1 Y$. Hence:

$$\begin{aligned}\Theta_{(\mathcal{L}+\lambda\circ\Phi)\omega} &= \theta \bar{\wedge} \Omega_{(\mathcal{L}+\lambda\circ\Phi)\omega} + (\mathcal{L} + \lambda \circ \Phi)\omega \\ &= \theta \bar{\wedge} \Omega_{\mathcal{L}\omega} + \mathcal{L}\omega + \lambda \circ (\theta \bar{\wedge} \Omega_{\Phi\omega} + \Phi\omega) = \Theta_{\mathcal{L}\omega} + \lambda \circ \Theta_{\Phi\omega}\end{aligned}$$

and, taking into account the notation we use for the pull-back of the differential forms, this is precisely $\psi^* \widehat{\Theta}$. □

At this point, we are ready to study the question of regularity for the constrained variational problems of the present work, which we shall consider before closing this section.

First of all, it must be noted that the unconstrained variational problem with Lagrangian density $(\mathcal{L} + \lambda \circ \Phi)\omega$ on $J^1(Y \times_Y E^*)$ is not regular in the ordinary sense, as the Hessian of the Lagrangian $\mathcal{L} + \lambda \circ \Phi$ contains zero rows due to the independence from first derivatives of λ. Intrinsically, this arises from the fact that the Cartan form $\Theta_{(\mathcal{L}+\lambda\circ\Phi)\omega}$ of this variational problem can be projected onto the form $\widehat{\Theta}$ on $J^1 Y \times_Y E^*$ which, as we saw, is also the projection of the Cartan form $\widetilde{\Theta}$. Taking into account Theorem 5.1 and inspired on the treatment given in [16,17] for the regularity of higher order variational problems whose Cartan form can be projected to lower order, we shall present the regularity question for the constrained variational problems we are considering, as follows:

We aim to obtain a condition on the constrained variational problem that will allow us to assure that for any section $\widehat{s} = (\overline{s}, \lambda) \in \Gamma(X, J^1Y \times_Y E^*)$, solution of Cartan equation $\widehat{s}^* i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = 0$, $\forall \widehat{D} \in \mathfrak{X}(S \times_Y E^*)$, its projection onto Y is a critical section s of the constrained variational problem such that $\overline{s} = j^1 s$ and $\lambda = \lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j^2 s)$.

As we will prove, this problem can be solved in a satisfactory way by means of the following regularity condition:

Definition 5.3. A constrained variational problem is regular when the polarity $\widehat{D} \in T_Y(S \times_Y E^*) \mapsto i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} \in \Lambda^n T^*(J^1Y \times_Y E^*)$, on the space $T_Y(S \times_Y E^*)$ of vector fields that are tangential to the submanifold $S \times_Y E^* \subset J^1Y \times_Y E^*$ and vertical over Y, is injective.

Proposition 5.4. *A constrained variational problem is regular if and only if along the submanifold $S \times_Y E^* \subset J^1Y \times_Y E^*$ it holds*:

$$\det \begin{pmatrix} \dfrac{\partial^2(\mathcal{L} + \lambda \circ \Phi)}{\partial y^i_\mu \partial y^j_\nu} & \dfrac{\partial \phi^\alpha}{\partial y^i_\mu} \\ \left(\dfrac{\partial \phi^\alpha}{\partial y^j_\nu}\right)^t & 0 \end{pmatrix} \neq 0 \tag{5.10}$$

Proof. Using flat connections associated to a system of local coordinates $(x^\nu, y^j, y^j_\nu, \lambda_\alpha)$ on $J^1Y \times E^*$ and taking into account formulas (2.2), (2.3), (3.1) and (3.2), along the submanifold $S \times_Y E^*$ we get:

$$\begin{aligned} \mathrm{d}\widehat{\Theta} &= \mathrm{d}\,(\theta \wedge (\Omega_{\mathcal{L}\omega} + \lambda \circ \Omega_{\Phi\omega}) + (\mathcal{L} + \lambda \circ \Phi)\omega) \\ &= \mathrm{d}\left(\frac{\partial \widehat{\mathcal{L}}}{\partial y^j_\nu}\right) \wedge \theta^j \wedge \omega_\nu + \frac{\partial \widehat{\mathcal{L}}}{\partial y^j} \theta^j \wedge \omega \end{aligned}$$

where $\widehat{\mathcal{L}} = \mathcal{L} + \lambda \circ \Phi$, $\theta^j = \mathrm{d}y^j - y^j_\nu\, \mathrm{d}x^\nu$, and $\omega_\nu = i_{(\partial/\partial x^\nu)}\omega$.

Let $\widehat{D} = f^j_\nu(\partial/\partial y^j_\nu) + f_\alpha(\partial/\partial \lambda_\alpha)$ be an arbitrary vector field, vertical over Y and tangential to the submanifold $S \times_Y E^* \subset J^1Y \times_Y E^*$, that is, $f^j_\nu(\partial \phi^\alpha/\partial y^j_\nu) = 0$.

Computing now the inner product of $\widehat{D}$ with $\mathrm{d}\widehat{\Theta}$ one gets:

$$i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = \widehat{D}\left(\frac{\partial \widehat{\mathcal{L}}}{\partial y^j_\nu}\right) \theta^j \wedge \omega_\nu = \left(\frac{\partial^2 \widehat{\mathcal{L}}}{\partial y^i_\mu \partial y^j_\nu} f^i_\mu + \frac{\partial \phi^\alpha}{\partial y^j_\nu} f_\alpha\right) \theta^j \wedge \omega_\nu$$

so $i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = 0$ for some $\widehat{D} = f^j_\nu(\partial/\partial y^j_\nu) + f_\alpha(\partial/\partial \lambda_\alpha) \in T_Y(S \times_Y E^*)$ if and only if the homogeneous system of $mn + k$ linear equations in $mn + k$ unknowns (f^j_ν, f^α) holds:

$$\frac{\partial^2 \widehat{\mathcal{L}}}{\partial y^i_\mu \partial y^j_\nu} f^i_\mu + \frac{\partial \phi^\alpha}{\partial y^j_\nu} f_\alpha = 0, \qquad \frac{\partial \phi^\alpha}{\partial y^i_\mu} f^i_\mu = 0$$

Therefore, from Definition 5.3 of regularity, we conclude. □

From here we are ready to prove the following:

Theorem 5.5. *Let $(\mathcal{L}\omega, S \subset J^1Y)$ be a regular constrained variational problem.*
If $\widehat{s} = (\overline{s}, \lambda) \in \Gamma(X, S \times_Y E^*)$ is solution of the Cartan equation:

$$\widehat{s}^* i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = 0, \quad \forall \widehat{D} \in \mathfrak{X}(S \times_Y E^*) \tag{5.11}$$

then the projection $s \in \Gamma(X, Y)$ of $\overline{s}$ to Y is a critical section of the constrained variational problem such that $\overline{s} = j^1 s$ and $\lambda = \lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j^2 s)$.

Proof. Let $y^j = s^j(x^1, \ldots, x^n)$ and $y^j_\nu = s^j_\nu(x^1, \ldots, x^n)$ be the equations of the section $\overline{s}$ in a local coordinate system.

Taking the inner product of $\mathrm{d}\widehat{\Theta}$ with vector fields of the form $\widehat{D} = (0, f_\alpha(\partial/\partial\lambda_\alpha)) \in \mathfrak{X}(S \times_Y E^*)$, for arbitrary f_α, we obtain using the local formulas from the proof of Proposition 5.4:

$$0 = \widehat{s}^* i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = f_\alpha \frac{\partial \phi^\alpha}{\partial y^j_\nu} \left(\frac{\partial s^j}{\partial x^\nu} - s^j_\nu \right) \omega$$

therefore, due to the arbitrariness of f_α, we get:

$$\frac{\partial \phi^\alpha}{\partial y^j_\nu} \left(\frac{\partial s^j}{\partial x^\nu} - s^j_\nu \right) = 0$$

Taking the inner product of $\mathrm{d}\widehat{\Theta}$ with vector fields of the form $\widehat{D} = (f^j_\nu(\partial/\partial y^j_\nu), 0)$, where f^j_ν must satisfy $f^j_\nu(\partial\phi^\alpha/\partial y^j_\nu) = 0$ for $\widehat{D}$ to be tangential to $S \times_Y E^*$, we get:

$$0 = \widehat{s}^* i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = -f^j_\nu \frac{\partial^2 \widehat{\mathcal{L}}}{\partial y^i_\mu \partial y^j_\nu} \left(\frac{\partial s^i}{\partial x^\mu} - s^i_\mu \right) \omega$$

where $\widehat{\mathcal{L}} = \mathcal{L} + \lambda \circ \Phi$, and considering the arbitrariness of f^j_ν under the tangency condition $f^j_\nu(\partial\phi^\alpha/\partial y^j_\nu) = 0$, there exist functions $g_\alpha \in \mathcal{C}^\infty(X)$ satisfying:

$$-\frac{\partial^2 \widehat{\mathcal{L}}}{\partial y^i_\mu \partial y^j_\nu} \left(\frac{\partial s^i}{\partial x^\mu} - s^i_\mu \right) = g_\alpha \frac{\partial \phi^\alpha}{\partial y^j_\nu}$$

So, the functions $(\partial s^i/\partial x^\mu) - s^i_\mu$ and g_α satisfy the system of homogeneous linear equations:

$$\frac{\partial^2 (\mathcal{L} + \lambda \circ \Phi)}{\partial y^i_\mu \partial y^j_\nu} \left(\frac{\partial s^i}{\partial x^\mu} - s^i_\mu \right) + \frac{\partial \phi^\alpha}{\partial y^j_\nu} g_\alpha = 0, \quad \frac{\partial \phi^\alpha}{\partial y^i_\mu} \left(\frac{\partial s^i}{\partial x^\mu} - s^i_\mu \right) = 0$$

which, due to regularity condition and Proposition 5.4 has only the trivial solution, and in particular: $s^i_\mu = \partial s^i/\partial x^\mu$, that is, $\bar{s} = j^1 s$.

Using now formula (5.5) for $\mathrm{d}\widehat{\Theta}$ and taking the inner product with tangential vector fields $\widehat{D}$ of $S \times_Y E^*$ we get, taking into account that $\bar{s} = j^1 s$ and $\Phi(\bar{s}) = 0$:

$$
\begin{aligned}
0 &= \widehat{s}^* i_{\widehat{D}}\, \mathrm{d}\widehat{\Theta} = (j^1 s, \lambda)^* (\theta(\widehat{D}) \circ (\mathbb{E}_{\mathcal{L}\omega} + \lambda \mathbb{E}_{\Phi\omega} - \mathrm{d}\lambda \bar{\wedge} \Omega_{\Phi\omega})) \\
&= (\mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda \bar{\wedge} \Omega_{\Phi\omega}(s))(\bar{D}^{\mathrm{v}}_s) \otimes \omega
\end{aligned}
$$

where $\bar{D}^{\mathrm{v}}_s \in T_s(\Gamma_S(X, Y))$ is the vertical component along s of the projection $\bar{D}_s$ on Y of the vector field $\widehat{D}_{\widehat{s}}$, arbitrary in $T_s(\Gamma_S(X, Y))$ when $\widehat{D} \in \mathfrak{X}(S \times_Y E^*)$.

Therefore $\bar{D}^{\mathrm{v}}_s = P_s(D^{\mathrm{v}}_s)$ with arbitrary $D^{\mathrm{v}}_s \in \Gamma(X, s^* VY)$, and thus $\bar{\mathcal{E}}_s(P_s(D^{\mathrm{v}}_s)) = 0$ for any $D^{\mathrm{v}}_s \in \Gamma(X, s^* VY)$, where we denote:

$$
\bar{\mathcal{E}}_s \otimes \omega = \mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda \bar{\wedge} \Omega_{\Phi\omega}(s)
$$

Following (3.9) we have now:

$$
P^+_s \bar{\mathcal{E}}_s \otimes \omega = \bar{\mathcal{E}}_s \otimes \omega + \lambda_{\bar{\mathcal{E}}_s} \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda_{\bar{\mathcal{E}}_s} \bar{\wedge} \Omega_{\Phi\omega}(s)
$$

where we may compute $\lambda_{\bar{\mathcal{E}}_s} = -\bar{\mathcal{E}}_s \circ N_s$ using (3.6) as follows:

$$
\begin{aligned}
\lambda_{\bar{\mathcal{E}}_s} \otimes \omega &= -\bar{\mathcal{E}}_s \circ N_s \otimes \omega = -(\mathcal{E}_{\mathcal{L}\omega}(s) \circ N_s) \otimes \omega - \lambda \circ (\mathcal{E}_{\Phi\omega}(s) \circ N_s) \otimes \omega \\
&\quad + \mathrm{d}\lambda \bar{\wedge} (\Omega_{\Phi\omega}(s) \circ N_s) = (\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \otimes \omega
\end{aligned}
$$

hence:

$$
\begin{aligned}
P^+_s \bar{\mathcal{E}}_s \otimes \omega &= \bar{\mathcal{E}}_s \otimes \omega + (\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}(\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \bar{\wedge} \Omega_{\Phi\omega}(s) \\
&= \mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \circ \mathcal{E}_{\Phi\omega}(s) \otimes \omega - \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} \bar{\wedge} \Omega_{\Phi\omega}(s) \\
&= P^+_s \mathcal{E}_{\mathcal{L}\omega}(s) \otimes \omega
\end{aligned}
$$

and finally by (3.10):

$$
\begin{aligned}
0 &= \bar{\mathcal{E}}_s(P_s(D^{\mathrm{v}}_s)) \otimes \omega = (P^+_s \bar{\mathcal{E}}_s)(D^{\mathrm{v}}_s)\omega + \mathrm{d}(\lambda_{\bar{\mathcal{E}}_s} \circ \Omega_{\Phi\omega}(s)(D^{\mathrm{v}}_s)) \\
&= (P^+_s \mathcal{E}_{\mathcal{L}\omega}(s))(D^{\mathrm{v}}_s) + \mathrm{d}[(\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \circ \Omega_{\Phi\omega}(s)(D^{\mathrm{v}}_s)], \quad \forall D^{\mathrm{v}}_s \in \Gamma(X, s^* VY)
\end{aligned}
$$

Taking now in particular sections D^{v}_s with compact support we get $\int_X (P^+_s \mathcal{E}_{\mathcal{L}\omega}(s))(D^{\mathrm{v}}_s) = 0$, so that $P^+_s \mathcal{E}_{\mathcal{L}\omega}(s) = 0$, that is, s is critical for the constrained variational problem, and $\mathrm{d}[(\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \circ \Omega_{\Phi\omega}(s)(D^{\mathrm{v}}_s)] = 0$ for any $D^{\mathrm{v}}_s \in \Gamma(X, s^* VY)$.

Taking in the latter arbitrary functions $f \in C^\infty(X)$ and sections $D^{\mathrm{v}}_s \in \Gamma(X, s^* VY)$ we get:

$$
0 = \mathrm{d}[(\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \circ \Omega_{\Phi\omega}(s)(f D^{\mathrm{v}}_s)] = \mathrm{d}f \wedge [(\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \circ \Omega_{\Phi\omega}(s)(D^{\mathrm{v}}_s)]
$$

and in virtue of the arbitrariness of f and D^{v}_s we obtain $(\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda) \circ \Omega_{\Phi\omega}(s) = 0$, which by Hypothesis (HY1) of Section 3 leads to $\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)} - \lambda = 0$, concluding the proof. □

The relevance of this result can be seen in the fact that it proves, together with the second part of Theorem 5.1, that for regular constrained variational problems the lifting

$$i : s \in \Gamma_S(X, Y) \mapsto (j^1 s, \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)}) \in \Gamma(X, S \times_Y E^*) \tag{5.12}$$

defines a bijective mapping between the set of critical sections of the constrained variational problem and the set of solutions of Cartan equation (5.11).

This justifies the consideration of the fibrations $S \times_Y E^* \xrightarrow{\pi} Y \xrightarrow{p} X$ (where $\dim(S \times_Y E^*) = \dim J^1 Y$), together with the $(n+1)$-form $\widehat{\Omega}_2 = \mathrm{d}\widehat{\Theta}|_{S \times_Y E^*}$ as the basic structure to construct the multi-symplectic formalism of the constrained variational calculus. In particular, for unconstrained problems, where $S = J^1 Y$, $E = 0$ and $\Phi = 0$, we get $S \times_Y E^* = J^1 Y$ and $\widehat{\Theta} = \Theta_{\mathcal{L}\omega}$, thus recovering the ordinary multi-symplectic formalism.

6. Examples

In this section we shall illustrate the general theory with two kinds of examples: problems with one independent variable (mechanics) and two cases with several independent variables, of a physical and geometrical interest respectively: general relativity in the sense of Palatini and a certain class of isoperimetric problems for hypersurfaces in a Riemannian manifold.

6.1. Mechanics

In this case we shall deal with problems in one independent variable, $X = \mathbb{R}$, where the coordinate t of $\mathbb{R}$ represents the "time" variable. The bundle Y will be in this case $M \times \mathbb{R}$, where M is a m-dimensional manifold with local coordinates (q^i), which shall be interpreted as the configuration space of a mechanical system with m degrees of freedom. Therefore we have $J^1(M \times \mathbb{R}) = TM \times \mathbb{R}$, where the tangent bundle TM, which has the induced local coordinate system $(q^i, \dot{q}^i)$, is called the "space of velocities" of the system.

Let $\mathcal{L}\,\mathrm{d}t$ be a Lagrangian density and S a constraint submanifold in $TM \times \mathbb{R}$ satisfying Hypotheses (HY1) and (HY2) from Section 3 with respect to a given k-rank vector bundle on $M \times \mathbb{R}$. The corresponding theory in this case is the so-called vakonomic mechanics (Arnold 1988), which has attracted much attention in the last years.

A typical example in this situation is that of linear constraints $S = \Delta \times \mathbb{R}$, where Δ is a $(m-k)$-dimensional non integrable distribution, E is the pull-back to $M \times \mathbb{R} \to M$ of the quotient TM/Δ and $\Phi : TM \times \mathbb{R} \to E$ is the bundle morphism induced by the canonical projection $TM \to TM/\Delta$. The application in this case of our general theory is instructive, as we may interpret it in terms of the geometry of the distribution $\Delta \subset TM$, and recover in this way from a non-linear perspective many of the classical results of the linear non-holonomic systems.

If we follow the different questions under consideration for the general theory, Section 3 delivers firstly the equations of vakonomic mechanics:

$$\Phi(s) = 0, \quad \mathcal{E}_{\mathcal{L}\,\mathrm{d}t}(s) \otimes \mathrm{d}t + \lambda(s) \circ \mathcal{E}_{\Phi\,\mathrm{d}t}(s) \otimes \mathrm{d}t - \mathrm{d}\lambda \overline{\wedge} \Omega_{\Phi\,\mathrm{d}t}(s) = 0 \tag{6.1}$$

where the multiplier $\lambda(s) \in \mathcal{C}^\infty(\mathbb{R})$ is univocally determined by formula:

$$\lambda(s) = \lambda_{\mathcal{E}_{\mathcal{L}\,\mathrm{d}t}}(s) = (-\mathcal{E}_{\mathcal{L}\,\mathrm{d}t} \circ N)(s)$$

where N is a solution of the system (3.6) along $j^2 s$.

The Cartan form $\widetilde{\Theta}$ from Section 4 is in this case a 1-form on $J^2(M \times \mathbb{R}) = T^{(2)}M \times \mathbb{R}$ (where $T^{(2)}M$ is the second order tangent bundle of M), which is projected through the morphism (5.1) into the 1-form $\widehat{\Theta} = \Theta_{\mathcal{L}\,\mathrm{d}t} + \lambda \circ \Theta_{\Phi\,\mathrm{d}t}$ on $J^1Y \times_Y E^* = (TM \times \mathbb{R}) \times_{M\times\mathbb{R}} E^*$ (Section 5). Specially interesting are the conservative (or autonomous) systems, which are those for which $\partial/\partial t$ is an infinitesimal symmetry in the sense of Definition 4.4. The corresponding Noether invariant $\widetilde{H} = -i_{(\partial/\partial t)}\widetilde{\Theta} \in \mathcal{C}^\infty(T^{(2)}M \times \mathbb{R})$, which can be projected via (5.1) to the function $\widehat{H} = -i_{(\partial/\partial t)}\widehat{\Theta} \in \mathcal{C}^\infty((TM \times \mathbb{R}) \times_{M\times\mathbb{R}} E^*)$, is a first integral for the equations of movement (6.1), called *energy*.

Regarding the regularity (Section 5), it is easy to see, applying Proposition 5.4 and the theorem of the inverse function, that $\widehat{\Theta}$ defines on the $(2m + 1)$-dimensional manifold $S \times_{M\times\mathbb{R}} E^*$ a contact 1-form whose local expression is:

$$\widehat{\Theta} = \widehat{p}_i\,\mathrm{d}q^i - \widehat{H}\,\mathrm{d}t$$

where the momenta $\hat{p}_i \in \mathcal{C}^\infty(S \times_{M\times\mathbb{R}} E^*)$ are given by:

$$\widehat{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \lambda_\alpha \frac{\partial \phi^\alpha}{\partial \dot{q}^i} \tag{6.2}$$

In this case (5.12) gives a canonical bijective correspondence between critical sections of the constrained variational problem and the integral curves of the characteristic vector field $\widehat{D}$ of the contact 1-form $\widehat{\Theta}$, that is, the only vector field $\widehat{D} \in \mathfrak{X}(S \times_{M\times\mathbb{R}} E^*)$ with $i_{\widehat{D}}\,\mathrm{d}\widehat{\Theta} = 0$, $\widehat{D}(t) = 1$.

In particular, for conservative systems one obtains the corresponding Hamiltonian formalism taking on the $2m$-dimensional manifold $M_{2m} = (S \times_{M\times\mathbb{R}} E^*)_{t=0}$ the symplectic metric $\Omega_2 = \mathrm{d}\widehat{\Theta}|_{M_{2m}}$ and the Hamiltonian $\widehat{H}|_{M_{2m}}$ (see [2] for a local version of this result).

In the following we shall illustrate the results above with three classical examples taking special emphasis on the solvability of equations (3.6), which constitute the fundamentals of this work.

6.1.1. The catenary

This is the mechanical system with two degrees of freedom, configuration space $M = \mathbb{R}^2$ coordinated by (x, y), Lagrangian $\mathcal{L} : TM \times \mathbb{R} \to \mathbb{R}$ given by $\mathcal{L} = y$ and (non-linear) constraint $S \subset TM \times \mathbb{R}$ defined by $\sqrt{\dot{x}^2 + \dot{y}^2} - 1 = 0$.

Taking as bundle $q : E \to M \times \mathbb{R}$ the direct product $(M \times \mathbb{R}) \times \mathbb{R}$, and the morphism $\Phi : TM \times \mathbb{R} \to E$ defined by the function $\Phi = \sqrt{\dot{x}^2 + \dot{y}^2} - 1$, for $\dot{x}^2 + \dot{y}^2 \neq 0$ we get the following momentum form $\Omega_{\Phi\,\mathrm{d}t}$ and Euler–Lagrange operator $\mathcal{E}_{\Phi\,\mathrm{d}t}$ associated to the constraint:

$$\Omega_{\Phi\,\mathrm{d}t} = \frac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}}(\dot{x}\,\mathrm{d}x + \dot{y}\,\mathrm{d}y), \quad \mathcal{E}_{\Phi\,\mathrm{d}t} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{x}^2)^{3/2}}(\dot{y}\,\mathrm{d}x - \dot{x}\,\mathrm{d}y)$$

The system of equations (3.6) has, on the dense open subset defined by $\dot{x}\ddot{y} - \dot{y}\ddot{x} \neq 0$, the only solution $N \in \Gamma(T^{(2)}M \times \mathbb{R}, (E^* \otimes TM)_{T^{(2)}M\times\mathbb{R}})$:

$$N = \frac{(\dot{x}^2 + \dot{y}^2)^{1/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \left(\dot{y}\frac{\partial}{\partial x} - \dot{x}\frac{\partial}{\partial y} \right)$$

Hence we get on this open subset the multiplier:

$$\lambda_{\mathcal{E}_{\mathcal{L}\,\mathrm{d}t}} = -\mathcal{E}_{\mathcal{L}\,\mathrm{d}t} \circ N = \frac{\dot{x}(\dot{x}^2 + \dot{y}^2)^{1/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \tag{6.3}$$

Applying now (5.2) we get Cartan's 1-form:

$$\widehat{\Theta} = \frac{\lambda}{(\dot{x}^2 + \dot{y}^2)^{1/2}}(\dot{x}\,\mathrm{d}x + \dot{y}\,\mathrm{d}y) + (y - \lambda)\,\mathrm{d}t$$

and, substituting λ by the expression (6.3), we also obtain the 1-form $\widetilde{\Theta}$.

As $\mathcal{L}$ and Φ are independent of time, the system is conservative and its energy is the function $\widehat{H} = -i_{\partial/\partial t}\widehat{\Theta} = \lambda - y$.

Last, the determinant (5.10) along $S \times_{M\times\mathbb{R}} E^*$ is $-\lambda$, so that for $\lambda \neq 0$ the catenary problem is a regular constrained variational problem. The momenta $\widehat{p}_x$, $\widehat{p}_y$ defined on $S \times_{M\times\mathbb{R}} E^*$ by formulas (6.2) are, respectively, $\lambda\dot{x}$ and $\lambda\dot{y}$, thus obtaining a Hamiltonian formulation for the problem on $M_4 = (S \times_{M\times\mathbb{R}} E^*)_{t=0}$, with symplectic 2-form $\Omega_2 = \mathrm{d}\widehat{p}_x \wedge \mathrm{d}x + \mathrm{d}\widehat{p}_y \wedge \mathrm{d}y$ and Hamiltonian $\widehat{H} = \sqrt{\widehat{p}_x^2 + \widehat{p}_y^2} - y$.

6.1.2. The soap film

In this case $M = \mathbb{R}^2$, coordinated by (y, v), and $\mathcal{L} = 2\pi y(1 + \dot{y}^2)^{1/2}$. The constraint S is given by the affine equation $\dot{v} - \pi y^2 = 0$. Taking again as bundle E the direct product $(M \times \mathbb{R}) \times \mathbb{R}$ and the morphism Φ defined by the function $\Phi = \dot{v} - \pi y^2$, one gets: $\Omega_{\Phi\,\mathrm{d}t} = \mathrm{d}v$ and $\mathcal{E}_{\Phi\,\mathrm{d}t} = -2\pi y\,\mathrm{d}y$, so that the system (3.6) has for $y \neq 0$ the only solution $N = -(1/2\pi y)(\partial/\partial y)$. Therefore on the dense open subset $y \neq 0$ one obtains the multiplier:

$$\lambda_{\mathcal{E}_{\mathcal{L}\,\mathrm{d}t}} = -\mathcal{E}_{\mathcal{L}\,\mathrm{d}t} \circ N = \frac{1 + \dot{y}^2 - y\ddot{y}}{y(1 + \dot{y}^2)^{3/2}} \tag{6.4}$$

From (5.2) we obtain the following Cartan 1-form:

$$\widehat{\Theta} = \frac{2\pi y\dot{y}}{\sqrt{1 + \dot{y}^2}}\,\mathrm{d}y + \lambda\,\mathrm{d}v + \left(\frac{2\pi y}{\sqrt{1 + \dot{y}^2}} - \pi y^2\lambda \right)\mathrm{d}t$$

and from here we also obtain $\widetilde{\Theta}$ substituting λ by the expression (6.4).

This system is also conservative, and its energy is $\widehat{H} = -i_{\partial/\partial t}\widehat{\Theta} = (-2\pi y/\sqrt{1 + \dot{y}^2}) + \pi y^2\lambda$. It is also regular, as the determinant (5.10) is $-2\pi y/(1 + \dot{y}^2)^{3/2} \neq 0$ on the dense open subset we are considering. Regarding the momenta $\widehat{p}_y$ and $\widehat{p}_v$ defined by (6.2), they are respectively $2\pi y\dot{y}/\sqrt{1 + \dot{y}^2}$ and λ, which together with the constraint equation allow us to obtain a Hamiltonian formulation on $M_4 = (S \times_{M\times\mathbb{R}} E^*)_{t=0}$ with symplectic 2-form $\Omega_2 = \mathrm{d}\widehat{p}_y \wedge \mathrm{d}y + \mathrm{d}\widehat{p}_v \wedge \mathrm{d}v$ and Hamiltonian $\widehat{H} = -\sqrt{4\pi^2 y^2 - \widehat{p}_y^2} + \pi y^2 \widehat{p}_v$.

6.1.3. The skateboard on an inclined plane

The configuration space of this problem is $M = \mathbb{R}^2_{(x,y)} \times S^1_\varphi$ where x and y represent the position of the skateboard on the plane and φ its angle with respect to the x-axis. The Lagrangian $\mathcal{L} : TM \times \mathbb{R} \to \mathbb{R}$ is the function $\mathcal{L} = (1/2)m(\dot{x}^2 + \dot{y}^2) + (1/2)I\dot{\varphi}^2 - \mathsf{g}y$ (g = gravity), and the constraint $S \subset TM \times \mathbb{R}$ is the linear submanifold defined by $\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$. Taking again as bundle $q : E \to M \times \mathbb{R}$ the direct product $(M \times \mathbb{R}) \times \mathbb{R}$ and the morphism $\Phi : TM \times \mathbb{R} \to E$ defined by the function $\Phi = \dot{x}\sin\varphi - \dot{y}\cos\varphi$ we obtain the following momentum form $\Omega_{\Phi\,\mathrm{d}t}$ and Euler–Lagrange operator $\mathcal{E}_{\Phi\,\mathrm{d}t}$ associated to the constraint:

$$\Omega_{\Phi\,\mathrm{d}t} = \sin\varphi\,\mathrm{d}x - \cos\varphi\,\mathrm{d}y$$
$$\mathcal{E}_{\Phi\,\mathrm{d}t} = -\dot{\varphi}\cos\varphi\,\mathrm{d}x - \dot{\varphi}\sin\varphi\,\mathrm{d}y + (\dot{x}\cos\varphi + \dot{y}\sin\varphi)\,\mathrm{d}\varphi$$

The system of equations (3.6) has solutions for $\dot{x}\cos\varphi + \dot{y}\sin\varphi \neq 0$ given by the affine subspace of $\Gamma(TM \times \mathbb{R}, (E^* \otimes TM)_{TM\times\mathbb{R}})$:

$$N = \left(\cos\varphi(\dot{x}\cos\varphi + \dot{y}\sin\varphi)\frac{\partial}{\partial x} + \sin\varphi(\dot{x}\cos\varphi + \dot{y}\sin\varphi)\frac{\partial}{\partial y} + \dot{\varphi}\frac{\partial}{\partial\varphi}\right) f + \frac{1}{\dot{x}\cos\varphi + \dot{y}\sin\varphi}\frac{\partial}{\partial\varphi}, \quad f \in \mathcal{C}^\infty(TM \times \mathbb{R}) \tag{6.5}$$

obtaining in this way a family of multipliers:

$$\lambda_{\mathcal{E}_{\mathcal{L}\,\mathrm{d}t}} = -\mathcal{E}_{\mathcal{L}\,\mathrm{d}t} \circ N = (m\ddot{x}\cos\varphi + (\mathsf{g} + m\ddot{y})\sin\varphi)(\dot{x}\cos\varphi + \dot{y}\sin\varphi) f + I\ddot{\varphi}\dot{\varphi} f + \frac{I\ddot{\varphi}}{\dot{x}\cos\varphi + \dot{y}\sin\varphi}, \quad f \in \mathcal{C}^\infty(TM \times \mathbb{R}) \tag{6.6}$$

Applying (5.2) we obtain Cartan's 1-form:

$$\widehat{\Theta} = (m\dot{x} + \lambda\sin\varphi)\,\mathrm{d}x + (m\dot{y} - \lambda\cos\varphi)\,\mathrm{d}y + I\dot{\varphi}\,\mathrm{d}\varphi - \left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2 + \mathsf{g}y\right)\mathrm{d}t$$

and from this, we may obtain a family of Cartan 1-forms $\widetilde{\Theta}$ substituting λ by the expressions (6.6)

As for the previous examples this is a conservative system with energy:

$$\widehat{H} = -i_{\partial/\partial t}\widehat{\Theta} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\varphi}^2 + \mathsf{g}y$$

Regarding the regularity, the determinant (5.10) is $-mI \neq 0$, so that our system is also regular. Eq. (6.2) give us the following momenta:

$$\widehat{p}_x = m\dot{x} + \lambda\sin\varphi, \quad \widehat{p}_y = m\dot{y} - \lambda\cos\varphi, \quad \widehat{p}_\varphi = I\dot{\varphi}$$

which, together with the constraint equation allow us to obtain a Hamiltonian formulation for this problem on $M_6 = (S \times_{M\times\mathbb{R}} E^*)_{t=0}$ with symplectic 2-form $\Omega_2 = \mathrm{d}\widehat{p}_x \wedge \mathrm{d}x + \mathrm{d}\widehat{p}_y \wedge \mathrm{d}y + \mathrm{d}\widehat{p}_\varphi \wedge \mathrm{d}\varphi$ and Hamiltonian:

$$\widehat{H} = \frac{1}{2m}\left(\widehat{p}_x \cos\varphi + \widehat{p}_y \sin\varphi\right)^2 + \frac{1}{2I}\widehat{p}_\varphi^2 + \mathsf{g}y$$

6.2. General relativity as a first order constrained variational problem

Following Palatini (1919) the starting point for the study of general relativity as a first order variational problem is the fibre product $\mathcal{M} \times_X \mathcal{C}$ over a four-dimensional manifold X oriented by a volume element ω (the space-time) of the bundle $\rho : \mathcal{M} \to X$ of Lorentz metrics on X and the affine bundle $\varrho : \mathcal{C} \to X$ (with associated vector bundle $S^2T^*X \otimes TX$) of symmetric linear connections on X.

On the bundle $J^1(\mathcal{M} \times_X \mathcal{C})$ we consider the constrained variational problem whose Lagrangian density is the scalar curvature associated to a metric $g \in \Gamma(X, \mathcal{M})$ and to a linear connection $\nabla \in \Gamma(X, \mathcal{C})$:

$$\mathcal{L}\omega(j_x^1(g, \nabla)) = (\mathrm{trace}(g^{-1} \cdot \mathrm{Curv}\nabla)\omega_g)_x$$

($\mathrm{Curv}\nabla =$ three-covariant, 1-contravariant curvature tensor of ∇, and $\omega_g =$volume element associated to g) and whose constraint submanifold is:

$$S = \{j_x^1(g, \nabla)/\nabla(x) = \nabla_g(x) = \text{Levi-Civita connection of } g\} \subset J^1(\mathcal{M} \times_X \mathcal{C})$$

In a local coordinate system $(x^\nu, g_{\alpha\beta}, \gamma^\sigma_{\alpha\beta}, g_{\alpha\beta,\rho}, \gamma^\sigma_{\alpha\beta,\rho})_{\alpha\le\beta}$ of $J^1(\mathcal{M} \times_X \mathcal{C})$, with $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, we have the following expression for the Lagrangian density:

$$\mathcal{L}\omega = R^\sigma_{\sigma\mu\nu} g^{\mu\nu} \sqrt{-\det g}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \qquad (6.7)$$

where $R^\tau_{\sigma\mu\nu} = \gamma^\tau_{\mu\nu,\sigma} - \gamma^\tau_{\sigma\nu,\mu} + \gamma^\alpha_{\mu\nu}\gamma^\tau_{\alpha\sigma} - \gamma^\alpha_{\sigma\nu}\gamma^\tau_{\alpha\mu}$ and $(g^{\mu\nu}) = g^{-1}(\mathrm{d}x^\mu, \mathrm{d}x^\nu)$, and the following equations for the constraint submanifold:

$$\phi^\sigma_{\mu\nu} = \left(\gamma^\sigma_{\mu\nu} - \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})\right)\sqrt{-\det g} = 0$$

Taking in this case as bundle E the pull-back to $\mathcal{M} \times_X \mathcal{C}$ of the vector bundle $S^2T^*X \otimes TX$, the constraint submanifold is $S = \Phi^{-1}(0_E)$, where $\Phi : J^1(\mathcal{M} \times_X \mathcal{C}) \to E$ is the bundle morphism defined by $\Phi(j_x^1(g, \nabla))\omega = (\nabla(x) - \nabla_g(x)) \otimes \omega_g$, whose local expression is given by the functions $\phi^\sigma_{\mu\nu}$ defined above.

Taking into account the canonical identification:

$$V(\mathcal{M} \times_X \mathcal{C})^* \otimes E = (V\mathcal{M}^* \oplus_{\mathcal{M}\times_X\mathcal{C}} V\mathcal{C}^*) \otimes E = [(S^2TX \otimes S^2T^*X \otimes TX) \oplus_X \times (S^2TX \otimes T^*X \otimes S^2T^*X \otimes TX)]_{\mathcal{M}\times_X\mathcal{C}}$$

simple computations prove that the momentum form $\Omega_{\Phi\omega}$ and Euler–Lagrange operator $\mathcal{E}_{\Phi\omega}$ associated to the E-valued Lagrangian density $\Phi\omega$ have the local expressions:

$$\Omega_{\Phi\omega} = \sum_{\mu\leq\nu} \frac{-1}{2} g^{\sigma\rho}(\omega_\nu \otimes \mathrm{d}g_{\rho\mu} + \omega_\mu \otimes \mathrm{d}g_{\nu\rho} - \omega_\rho \otimes \mathrm{d}g_{\mu\nu}) \otimes \left(\mathrm{d}x^\mu\, \mathrm{d}x^\nu \otimes \frac{\partial}{\partial x^\sigma}\right) \tag{6.8}$$

$$\mathcal{E}_{\Phi\omega} \otimes \omega = \left[(\mathcal{E}_{\Phi\omega})_{\mathcal{M}} \otimes \omega, \sum_{\alpha\leq\beta} \mathrm{d}\gamma^\nu_{\alpha\beta} \otimes \left(\mathrm{d}x^\alpha\, \mathrm{d}x^\beta \otimes \frac{\partial}{\partial x^\nu}\right) \otimes \omega_g\right] \tag{6.9}$$

where $\omega_\mu = i_{\partial/\partial x^\mu}\omega_g$, $\mathrm{d}x^\mu\, \mathrm{d}x^\nu$ is the symmetrization of the tensor product $\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ and $(\mathcal{E}_{\Phi\omega})_{\mathcal{M}}$ is the $V\mathcal{M}^* \otimes E$-component of $\mathcal{E}_{\Phi\omega}$.

Given a section $(g, \nabla) \in \Gamma(X, \mathcal{M} \times_X \mathcal{C})$, and taking (6.8) into account, we obtain that for a section $N = \sum_{\mu\leq\nu} N_{\mu\nu}(\partial/\partial g_{\mu\nu}) + \sum_{\alpha\leq\beta} N^\nu_{\alpha\beta}(\partial/\partial\gamma^\nu_{\alpha\beta})$ of $(g, \nabla)^* V(\mathcal{M} \times_X \mathcal{C})$ to be incident with $\Omega_{\Phi\omega}(g, \nabla)$, there must hold that for any indices σ, μ, ν:

$$N_{\rho\mu} g^{\sigma\rho}\omega_\nu + N_{\nu\rho} g^{\sigma\rho}\omega_\mu - N_{\mu\nu} g^{\sigma\rho}\omega_\rho = 0$$

where $N_{\alpha\beta}$ is defined as $N_{\beta\alpha}$ for $\alpha > \beta$, therefore:

$$N_{\sigma\mu}\frac{\partial}{\partial x^\nu} + N_{\nu\sigma}\frac{\partial}{\partial x^\mu} - N_{\mu\nu}\frac{\partial}{\partial x^\sigma} = 0, \quad N_{\mu\nu}\frac{\partial}{\partial x^\sigma} + N_{\sigma\mu}\frac{\partial}{\partial x^\nu} - N_{\nu\sigma}\frac{\partial}{\partial x^\mu} = 0$$

and hence $N_{\sigma\mu}(\partial/\partial x^\nu) = 0$, that is: $N_{\sigma\mu} = 0$.

This proves, in particular, taking the 1-jets at any point, that the matrix $(\partial\phi^\sigma_{\mu\nu}/\partial g_{\alpha\beta,\rho})$ has constant rank 40 along S and therefore, that all the conditions in Hypothesis (HY1) are satisfied.

On the other hand, taking into account the expression (6.9) for $\mathcal{E}_{\Phi\omega} \otimes \omega$, the only section $N \in \Gamma(J^2(\mathcal{M} \times_X \mathcal{C}), V(\mathcal{M} \times_X \mathcal{C}) \otimes E^*)$ solution for the system (3.6) is given by:

$$N \otimes \omega_g = \left[0, \frac{\partial}{\partial\gamma^\nu_{\alpha\beta}} \otimes \left(\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} \otimes \mathrm{d}x^\nu\right) \otimes \omega\right] \tag{6.10}$$

In fact, except for the presence of the volume element ω, the second component in (6.10) is simply the pull-back to $J^2(\mathcal{M} \times_X \mathcal{C})$ of the identity section I of $E \otimes E^*$.

Therefore, our constraint submanifold $S = \Phi^{-1}(0_E)$ also satisfies Hypothesis (HY2) of the theory.

We now take into account (see, for example [32, p. 500]) that the Euler–Lagrange operator $\mathcal{E}_{\mathcal{L}\omega}$ of the Lagrangian density (6.7) as unconstrained variational problem is given by the expression:

$$\mathcal{E}_{\mathcal{L}\omega}(g, \nabla) \otimes \omega = [\mathrm{Eins}(g, \nabla)\omega_g, \mathrm{d}^\nabla(g^{-1} \otimes \omega_g) - \mathrm{Sym}(\mathrm{Id} \otimes \cdot\mathrm{d}^\nabla(g^{-1} \otimes \omega_g))] \tag{6.11}$$

where $\mathrm{Eins}(g, \nabla)$ is the Einstein tensor associated to the metric tensor g and to the linear connection ∇, d^∇ denotes the covariant derivative of $g^{-1} \otimes \omega_g$ with respect to ∇ and

$\cdot\mathrm{d}^\nabla(g^{-1}\otimes\omega_g)$ denotes the contraction of a contravariant index of $\mathrm{d}^\nabla(g^{-1}\otimes\omega_g)$ with the covariant index produced by d^∇. We obtain thus the following expression for $\lambda_{\mathcal{E}_{\mathcal{L}\omega}(g,\nabla)}$:

$$\lambda_{\mathcal{E}_{\mathcal{L}\omega}(g,\nabla)}\omega_g = -\langle\mathcal{E}_{\mathcal{L}\omega}(g,\nabla), N(g,\nabla)\otimes\omega_g\rangle = -\langle\mathcal{E}_{\mathcal{L}\omega}(g,\nabla),[0,\mathrm{I}\otimes\omega]\rangle$$
$$= [0, -\mathrm{d}^\nabla(g^{-1}\otimes\omega_g) + \mathrm{Sym}(\mathrm{Id}\otimes\cdot\mathrm{d}^\nabla(g^{-1}\otimes\omega_g))] \qquad (6.12)$$

From this result and from (6.8), (6.9) and (6.11) we obtain the following Euler–Lagrange operator for the constrained variational problem:

$$P^+_{(g,\nabla)}(\mathcal{E}_{\mathcal{L}\omega}(g,\nabla))\otimes\omega$$
$$= \mathcal{E}_{\mathcal{L}\omega}(g,\nabla)\otimes\omega + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(g,\nabla)}\circ\mathcal{E}_{\Phi\omega}(g,\nabla)\otimes\omega - \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}(g,\nabla)}\bar{\wedge}\Omega_{\Phi\omega}(g,\nabla)$$
$$= [\mathrm{Eins}(g,\nabla)\omega_g, \mathrm{d}^\nabla(g^{-1}\otimes\omega_g) - \mathrm{Sym}(\mathrm{Id}\cdot\mathrm{d}^\nabla(g^{-1}\otimes\omega_g))]$$
$$+[0, -\mathrm{d}^\nabla(g^{-1}\otimes\omega_g) + \mathrm{Sym}(\mathrm{Id}\cdot\mathrm{d}^\nabla(g^{-1}\otimes\omega_g))] = [\mathrm{Eins}(g,\nabla)\omega_g, 0]$$

Therefore, following Corollary 3.9, an admissible section (g,∇_g) is critical for the constrained variational problem if and only if $\mathrm{Eins}(g,\nabla_g)=0$.

The key point is the following: the constraint condition (that is, $\nabla=\nabla_g$) is in this case equivalent to the second group of Euler–Lagrange equations $\mathrm{d}^\nabla(g^{-1}\otimes\omega_g) - \mathrm{Sym}(\mathrm{Id}\cdot\mathrm{d}^\nabla(g^{-1}\otimes\omega_g)) = 0$ of the Lagrangian density as an unconstrained variational problem (see [32, p.502]), and hence the critical sections for the constrained and unconstrained variational problem are the same. However this only holds because of the particular choice of the Lagrangian (6.7). In general for other possible choices to establish the theory this circumstance would automatically disappear, producing then the constrained and unconstrained setup different variational problems.

The Cartan form of the constrained variational problem is:

$$\widetilde{\Theta} = \Theta_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}}\circ\Theta_{\Phi\omega}$$

where $\lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j^2_x(g,\nabla)) = \lambda_{\mathcal{E}_{\mathcal{L}\omega}(g,\nabla)}(x)$ is defined by (6.12) and where $\Theta_{\mathcal{L}\omega}$ and $\Theta_{\Phi\omega}$ are the 4-forms on $J^1(\mathcal{M}\times_X\mathcal{C})$:

$$\Theta_{\mathcal{L}\omega} = g^{\mu\nu}[\mathrm{d}\gamma^\sigma_{\mu\nu}\wedge\omega_\sigma - \mathrm{d}\gamma^\sigma_{\sigma\nu}\wedge\omega_\mu + (\gamma^\alpha_{\mu\nu}\gamma^\sigma_{\alpha\sigma} - \gamma^\alpha_{\sigma\nu}\gamma^\sigma_{\alpha\mu})\omega_g] \qquad (6.13)$$

$$\Theta_{\Phi\omega} = \sum_{\mu\le\nu}\left[\gamma^\sigma_{\mu\nu}\omega_g - \frac{1}{2}g^{\sigma\rho}(\mathrm{d}g_{\mu\rho}\wedge\omega_\nu + \mathrm{d}g_{\nu\rho}\wedge\omega_\mu - \mathrm{d}g_{\mu\nu}\wedge\omega_\rho)\right]$$
$$\otimes\left(\mathrm{d}x^\mu\,\mathrm{d}x^\nu\otimes\frac{\partial}{\partial x^\sigma}\right)$$

As the multiplier $\lambda_{\mathcal{E}_{\mathcal{L}\omega}}$ only depends on the first derivatives and vanishes on the constraint submanifold we conclude that $\widetilde{\Theta}$ is a four-form on $J^1(\mathcal{M}\times_X\mathcal{C})$ such that $\widetilde{\Theta}\Big|_S = \Theta_{\mathcal{L}\omega}$.

Considering Definition 5.3, in this case regularity does not hold, the tangential vector fields generated by $\partial/\partial\gamma^\sigma_{\mu\nu,\rho}$ are in the kernel of the corresponding polarity $D\in T_Y(S\times_Y E^*)\mapsto i_D\,\mathrm{d}\widehat{\Theta}$, which therefore is not injective.

Let us finish the study of this example by noting that the constrained variational problem under consideration is nothing else but the Lagrangian reduction of the usual metric formulation of general relativity by the bundle morphism $\tau : J^1\mathcal{M} \to \mathcal{M} \times_X \mathcal{C}$ defined by the rule $\tau(j_x^1 g) = (g(x), \nabla_g(x))$. Following the general framework of Lagrangian reduction (see, for example [10]), Hilbert's Lagrangian is, precisely, $\tau_{(1)}^* \mathcal{L}\omega$ where $\mathcal{L}\omega$ is Palatini's Lagrangian defined on $J^1(\mathcal{M} \times_X \mathcal{C})$ by formula (6.7) and $\tau_{(1)} : J^2\mathcal{M} \to J^1(\mathcal{M} \times_X \mathcal{C})$ is the 1-lifting of τ (that is, $\tau_{(1)}(j_x^2 g) = j_x^1(\tau \circ j^1 g)$). As we have seen in this example a metric tensor $g \in \Gamma(X, \mathcal{M})$ satisfies Einstein equations if and only if $\tau \circ j^1 g = (g, \nabla_g) \in \Gamma_S(X, \mathcal{M} \times_X \mathcal{C})$ is a critical section of the constrained variational problem of Palatini and conversely (reduction and reconstruction). Regarding the corresponding Cartan forms, following (6.13) $\Theta_{\mathcal{L}\omega}$ can be projected to $\mathcal{M} \times_X \mathcal{C}$ and $\Theta_{\tau_{(1)}^* \mathcal{L}\omega} = \tau^* \Theta_{\mathcal{L}\omega}$, thus recovering the well-known result that the Cartan form of Hilbert's Lagrangian, which in principle should be defined on $J^3\mathcal{M}$, can be projected to $J^1\mathcal{M}$.

6.3. Isoperimetric problems for hypersurfaces in a Riemannian manifold

In this case we shall consider immersions of a given n-dimensional manifold X into a Riemannian manifold M with metric tensor $\overline{g}$.

Let $p : Y = X \times M \to X$ be the trivial bundle defined by the product of both manifolds and identify its sections with mappings $s : X \to M$. Given a fixed volume element ω on X, we shall consider the constrained variational problem on J^1Y whose constraint submanifold is:

$$S = \{j_x^1 s / \omega_g = \omega\} \subset J^1 Y$$

where ω_g is the volume element induced on X by the first fundamental form of the hypersurface $g = s^*\overline{g}$, and ω is the fixed volume element on X. This is the isoperimetric constraint, according to which admissible sections $s \in \Gamma_S(X, Y)$ are immersions that induce a fixed "area" element on the hypersurface $s(X) \subset M$. The Lagrangian densities we shall consider for our constrained problem will be of the form:

$$\mathcal{L}\omega(j_x^1 s) = s^*(i_D \omega_{\overline{g}}) \tag{6.14}$$

where $\omega_{\overline{g}}$ is the volume $(n+1)$-form on M induced by the metric tensor $\overline{g}$, and D is a fixed vector field on M. As can be seen by a simple application of Stokes' Theorem, in the case that $\mathrm{div}_{\overline{g}} D = 1$ and $s(X)$ is the boundary of a compact domain in M, this Lagrangian density represents the volume enclosed by the hypersurface.

Taking the bundle $E = Y \times \mathbb{R}$, the constraint submanifold is $S = \Phi^{-1}(0)$, where $\Phi : J^1 Y \to E$ is defined by $\Phi(j_x^1 s)\omega = \omega_g(x) - \omega(x)$. Considering local coordinate systems (x^ν) for X with $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, (y^j) for M with $\overline{g} = \overline{g}_{ij} \mathrm{d}y^i\, \mathrm{d}y^j$ and the induced coordinates (x^ν, y^j, y_ν^j) on J^1Y, the function defining the constraint is:

$$\phi = \sqrt{\det g} - 1 \tag{6.15}$$

where $g = (g_{\mu\nu})$ with $g_{\mu\nu} = y_\mu^i y_\nu^j \overline{g}_{ij}$. On the other hand if we have the local expression $D = q^j(\partial/\partial y^j)$ $(q^j \in \mathcal{C}^\infty(M))$, the Lagrangian density defined by (6.14) is:

$$\mathcal{L}\omega = (-1)^{j+1} q^j \sqrt{\det \bar{g}}\, \det(y^i_\nu)_{[j]}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{6.16}$$

where $\det(y^i_\nu)_{[j]}$ is the minor obtained by elimination of the jth row of the matrix (y^i_ν)

From (6.15) we may compute the associated momentum form $\Omega_{\Phi\omega}$ and Euler–Lagrange operator $\mathcal{E}_{\Phi\omega}$. Using the natural identification of VY with $X \times TM$, these are:

$$\Omega_{\Phi\omega} = P^\perp \cdot \omega_g, \quad \mathcal{E}_{\Phi\omega} \otimes \omega = -i_{\mathbf{H}}\bar{g} \otimes \omega_g$$

where $\mathbf{H}(j^2_x s)$ denotes the TM-valued function on J^2Y that produces the mean curvature vector associated to $s : X \to M$ at $s(x)$ (that is, the trace of the Weingarten endomorphism), $P^\perp$ denotes the $T^*M \otimes TX$-valued function on J^1Y defined at $j^1_x s$ as the orthogonal projection of $T_{s(x)}M$ to T_xX given by $s : X \to M$, and $P^\perp \cdot \omega_g$ denotes the contraction of its contravariant component with a covariant component of ω_g.

It is clear that $(\partial\phi/\partial y^i_\mu)$ vanishes only if $P^\perp(j^1_x s) = 0$, that is, at singular points with $\mathrm{Im}s_* = \{0\}$. Therefore the constraint satisfies Hypothesis (HY1). On the other hand, on a dense open subset of J^2Y there exists a unique solution of the system of linear equations (3.6), the section $N \in \Gamma(J^2Y, E^* \otimes VY) = \Gamma(J^2Y, TM)$ defined by:

$$N \otimes \omega_g = -\frac{\mathbf{H}}{\|\mathbf{H}\|^2} \otimes \omega$$

defined at those points $j^2_x s$ where the mean curvature $\|\mathbf{H}\|$ of the hypersurface does not vanish. Hence Hypothesis (HY2) also holds.

Lengthy but trivial computations lead us from (6.16) to the intrinsic and local expressions of the Euler–Lagrange operator associated to the Lagrangian density $\mathcal{L}\omega$ as a variational problem without constraints:

$$\begin{aligned}\mathcal{E}_{\mathcal{L}\omega} \otimes \omega &= \mathrm{d}y^k \otimes (-1)^{k+1} \det(y^i_\nu)_{[k]} \frac{\partial}{\partial y^j}(q^j \sqrt{\det \bar{g}}) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= (\mathrm{div}_{\bar{g}} D)\, \mathrm{d}y^k \otimes s^* i_{(\partial/\partial y^k)} \omega_{\bar{g}} = (\mathrm{div}_{\bar{g}} D) i_{\mathbf{N}} \bar{g} \otimes \omega_g\end{aligned} \tag{6.17}$$

where $\mathrm{div}_{\bar{g}} D$ stands for the divergence of the vector field D with respect to the volume element $\omega_{\bar{g}}$ and $\mathbf{N}(j^2_x s) \in T_{s(x)}M$ is the normal vector field associated to the hypersurface defined by s and to the chosen orientations $\omega_{\bar{g}}$ and ω at $s(x)$.

It must be noted that $\mathcal{E}_{\mathcal{L}\omega} \otimes \omega$ can be projected from J^2Y to J^1Y. This reflects the fact that even though $s^* i_D \omega_{\bar{g}}$ does depend on $j^1 s$, its differential does not depend on $j^2 s$, in fact $\mathrm{d}s^* i_D \omega_{\bar{g}} = s^*\, \mathrm{d}i_D \omega_{\bar{g}}$.

The Lagrange multiplier (3.14) is then:

$$\lambda_{\mathcal{E}_{\mathcal{L}\omega}} = -\mathcal{E}_{\mathcal{L}\omega} \circ N = (\mathrm{div}_{\bar{g}} D) \bar{g}\left(\mathbf{N}, \frac{\mathbf{H}}{\|\mathbf{H}\|^2}\right) = \frac{\mathrm{div}_{\bar{g}} D}{\|\mathbf{H}\|} \tag{6.18}$$

where the last equality holds if we assume that the orientations are chosen so that $\bar{g}(\mathbf{N}, \mathbf{H}) > 0$.

From this result we obtain the following expression for the Euler–Lagrange operator of the constrained variational problem:

$$P_s^{+}\mathcal{E}_{\mathcal{L}\omega}(s)\otimes\omega = \mathcal{E}_{\mathcal{L}\omega}(s)\otimes\omega + \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)}\mathcal{E}_{\Phi\omega}(s)\otimes\omega - \mathrm{d}\lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)}\bar{\wedge}\Omega_{\Phi}(s)$$
$$= (\mathrm{div}_{\bar{g}}D)i_{\mathbf{N}}\bar{g}\otimes\omega_g - \frac{\mathrm{div}_{\bar{g}}D}{\|\mathbf{H}\|}i_{\mathbf{H}}\bar{g}\otimes\omega_g - d\left(\frac{\mathrm{div}_{\bar{g}}D}{\|\mathbf{H}\|}\right)\bar{\wedge}(P^{\perp}\cdot\omega_g)$$
$$= -d\left(\frac{\mathrm{div}_{\bar{g}}D}{\|\mathbf{H}\|}\right)\bar{\wedge}(P^{\perp}\cdot\omega_g) = -i_{\mathrm{grad}_g\left(\frac{\mathrm{div}_{\bar{g}}D}{\|\mathbf{H}\|}\right)}\bar{g}\otimes\omega_g$$

where $\mathrm{grad}_g(\mathrm{div}_{\bar{g}}D/\|\mathbf{H}\|)$ stands for the gradient of the function $\mathrm{div}_{\bar{g}}D/\|\mathbf{H}\|$ on the hypersurface, with respect to its first fundamental form g, as a vector field on the ambient manifold M defined along $s(X)$, which coincides with the orthogonal projection to $s(X)$ of the gradient with respect to $\bar{g}$ of the function $\mathrm{div}_{\bar{g}}D/\|\mathbf{H}\|$.

Therefore, following Corollary 3.9, an admissible section s is critical for the constrained variational problem if and only if $\mathrm{grad}_g(\mathrm{div}_{\bar{g}}D/\|\mathbf{H}\|) = 0$. For the case $\mathrm{div}_{\bar{g}}D = 1$ (i.e., for the Lagrangian density that gives the enclosed volume) the solutions are those hypersurfaces with (non-vanishing) constant mean curvature $\|\mathbf{H}\|$.

The Cartan form of the constrained variational problem is:

$$\widetilde{\Theta} = \Theta_{\mathcal{L}\omega} + \lambda_{\mathcal{E}_{\mathcal{L}\omega}}\circ\Theta_{\Phi\omega}$$

where $\lambda_{\mathcal{E}_{\mathcal{L}\omega}}(j_x^2 s) = \lambda_{\mathcal{E}_{\mathcal{L}\omega}(s)}(x)$ is defined by (6.18) and where $\Theta_{\mathcal{L}\omega}$ and $\Theta_{\Phi\omega}$ are the n forms on $J^1Y = (T^*X\otimes_{X\times M}TM)$ whose intrinsic and local expressions are:

$$\begin{aligned}\Theta_{\mathcal{L}\omega}(j_x^1 s) &= \mathrm{d}y^j\wedge s^*(i_{(\partial/\partial y^j)}i_D\omega_{\bar{g}})(x) + (1-n)s^*(i_D\omega_{\bar{g}})(x)\\ &= (-1)^{k+1}q^k\sqrt{\det\bar{g}}[(1-n)\det(y_\nu^i)_{[k]}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\\ &\quad +\mathrm{sgn}(k-j)(-1)^{j+1}\det(y_\nu^i)_{[k,j]}^{[\mu]}\,\mathrm{d}y^j\wedge\mathrm{d}x^1\wedge\cdots^{[\mu]}\cdots\wedge\mathrm{d}x^n]\\ \Theta_{\Phi\omega}(j_x^1 s) &= P^{\perp}(s(x))\bar{\wedge}\omega_g(x) + (1-n)\omega_g(x) - \omega(x)\\ &= g^{\mu\nu}y_\nu^k\bar{g}_{kj}\sqrt{\det g}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^{\mu-1}\wedge\mathrm{d}y^j\wedge\mathrm{d}x^{\mu+1}\wedge\cdots\wedge\mathrm{d}x^n\\ &\quad +(1-n)\sqrt{\det g}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n - \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n\end{aligned}\tag{6.19}$$

where $\det(y_\nu^i)_{[k,j]}^{[\mu]}$ stands for the minor of (y_ν^i) corresponding to the elimination of the k, jth rows and μth column, and $(g^{\mu\nu})$ stands for the inverse matrix of $(g_{\mu\nu})$.

Any vector field $V\in\mathfrak{X}(M)$, infinitesimal symmetry for $\omega_{\bar{g}}$ and D (i.e. $L_V\omega_{\bar{g}} = 0$, $[V, D] = 0$) naturally induces a p-vertical symmetry of our constrained variational problem $\mathbf{V}\in\mathfrak{X}(Y)$. Applying Noether Theorem 4.5 for these symmetries and the intrinsic expressions in (6.19) we may compute the following Noether invariants:

$$(j^2s)^* i_{j^2\mathbf{V}}\widetilde{\Theta} = s^* i_V\left(i_{D+(\mathrm{div}_{\bar{g}}D/\|\mathbf{H}\|)\mathbf{N}}\omega_{\bar{g}}\right)$$

Thus obtaining non-trivial $(n-1)$-forms that are closed whenever the hypersurface defined by s is critical for our constrained variational problem.

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Appendix A. Independence of the theory with respect to the vector bundle E and the bundle morphism $\Phi : J^1Y \to E$ defining the constraint submanifold $S = \Phi^{-1}(0_E)$.

As can be seen, the theory is based on the consideration of a vector bundle $q : E \to Y$ and a bundle morphism $\Phi : J^1Y \to E$ on Y, which satisfy Hypotheses (HY1) and (HY2), and such that the constraint submanifold is given by $S = \Phi^{-1}(0_E)$. A natural question now is: Is the whole theory independent of the chosen vector bundle E and morphism Φ? In this appendix we shall give a precise answer proceeding as follows:

First of all, we shall impose Hypothesis (HY2) from Section 3 in the following way, more suitable for our purposes:

Hypothesis (HY2′). On a dense open subset of $S^{(2)} \subset J^2Y$ (the inverse image of $S \subset J^1Y$ by the canonical projection), there exists a section $N \in \Gamma(S^{(2)}, (E^* \otimes VY)_{J^2Y})$, solution of the system of equations:

$$\Omega_{\Phi\omega} \circ N = 0, \qquad \mathcal{E}_{\Phi\omega} \circ N = \mathrm{I} \tag{3.6′}$$

Locally:

$$\sum_{j=1}^{m} \frac{\partial \phi^\alpha}{\partial y^j_\nu} N^j_\beta = 0, \quad \sum_{j=1}^{m} \left[\frac{\partial \phi^\alpha}{\partial y^j} - \sum_\nu \frac{\partial}{\partial x^\nu} \left(\frac{\partial \phi^\alpha}{\partial y^j_\nu} \right) \right] N^j_\beta = \delta^\alpha_\beta \tag{3.7′}$$

along $S^{(2)}$, $1 \le \alpha, \beta \le k$, $1 \le \nu \le n$.

Let $\{N\}_{(E,\Phi)}$ be the set of solutions of the system of equations (3.6′). If $q' : E' \to Y$ and $\Phi' : J^1Y \to E'$ are another k-rank vector bundle and bundle morphism on Y satisfying Hypothesis (HY1), then due to this hypothesis there exists a unique vector bundle isomorphism $\tau : E_S \to E'_S$ between the vector bundles on S induced by E and E', such that with the usual identifications: $\mathrm{d}\Phi' = \tau \circ \mathrm{d}\Phi$ along S. Let $\tau^{(2)} : (E^* \otimes VY)_{S^{(2)}} \to \left(E'^* \otimes VY\right)_{S^{(2)}}$ be the isomorphism defined by the action of τ on $E^*_{S^{(2)}}$ and the identity morphism on $VY_{S^{(2)}}$. In this situation we have the following:

Lemma 1. *E' and Φ' satisfy* Hypothesis (HY2′) *and it holds*:

$$\{N'\}_{(E',\Phi')} = \tau^{(2)}\{N\}_{(E,\Phi)}$$

Proof. Given $N \in \{N\}_{(E,\Phi)}$, we shall see that $N' = \tau^{(2)}N$ is a solution for the system of linear equations $\Omega_{\Phi'\omega} \circ N = 0$, $\mathcal{E}_{\Phi' \circ N} = \mathrm{I}'$. In a local coordinate system, let $(a^{\alpha'}_\alpha)$, $a^{\alpha'}_\alpha \in \mathcal{C}^\infty(S)$, $1 \le \alpha, \alpha' \le k$ be the matrix of the isomorphism $\tau : E_S \to E'_S$ with respect to trivializations e_α, $e'_{\alpha'}$ of the bundles E and E'. The equation $\mathrm{d}\Phi' = \tau \circ \mathrm{d}\Phi$ along S can be locally expressed by $\mathrm{d}\phi'^{\alpha'} = a^{\alpha'}_{\alpha}\mathrm{d}\phi^{\alpha}$. On the other hand, if $N = N^j_\alpha e^{*\alpha} \otimes (\partial/\partial y^j)$ then $N' = \tau^{(2)}N = N'^j_{\alpha'} e'^{*\alpha'} \otimes (\partial/\partial y^j)$ where $N'^j_{\alpha'} = b^{\alpha}_{\alpha'} N^j_\alpha$ for $(b^{\alpha}_{\alpha'})$ the inverse matrix of $(a^{\alpha'}_{\alpha})$. Therefore:

$$(\Omega_{\Phi'\omega} \circ N')^{\beta'\nu}_{\alpha'} = \frac{\partial \phi'^{\beta'}}{\partial y^j_\nu} N'^j_{\alpha'} = a^{\beta'}_{\beta}\frac{\partial \phi^{\beta}}{\partial y^j_\nu} b^{\alpha}_{\alpha'} N^j_\alpha = a^{\beta'}_{\beta} b^{\alpha}_{\alpha'}\frac{\partial \phi^{\beta}}{\partial y^j_\nu} N^j_\alpha = 0$$

$$\begin{aligned}(\mathcal{E}_{\Phi'\omega} \circ N')^{\beta'}_{\alpha'} &= \left(\frac{\partial \phi'^{\beta'}}{\partial y^j} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \phi'^{\beta'}}{\partial y^j_\nu}\right)\right) N'^j_{\alpha'} = \left(a^{\beta'}_{\beta}\frac{\partial \phi^{\beta}}{\partial y^j} - \frac{\partial}{\partial x^\nu}\left(a^{\beta'}_{\beta}\frac{\partial \phi^{\beta}}{\partial y^j_\nu}\right)\right) b^{\alpha}_{\alpha'} N^j_\alpha \\ &= a^{\beta'}_{\beta} b^{\alpha}_{\alpha'}\left(\frac{\partial \phi^{\beta}}{\partial y^j} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \phi^{\beta}}{\partial y^j_\nu}\right)\right) N^j_\alpha - b^{\alpha}_{\alpha'}\frac{\partial a^{\beta'}_{\beta}}{\partial x^\nu}\frac{\partial \phi^{\beta}}{\partial y^j_\nu} N^j_\alpha = a^{\beta'}_{\beta} b^{\alpha}_{\alpha'}\delta^{\beta}_{\alpha} = \delta^{\beta'}_{\alpha'}\end{aligned}$$

Hence, along the dense open subset of $S^{(2)}$ where N is defined we get that $N' = \tau^{(2)}N$ is a solution for the system of equations $\Omega_{\Phi'\omega} \circ N' = 0$, $\mathcal{E}_{\Phi'\omega} \circ N' = \mathrm{I}'$, and therefore E' and Φ' satisfy Hypothesis (HY2′).

If $\{N'\}_{(E',\Phi')}$ is the corresponding set of solutions, using the same argument for the inverse isomorphism τ^{-1}, we get the identity $\{N'\}_{(E',\Phi')} = \tau^{(2)}\{N\}_{(E,\Phi)}$. □

Let now $\{P_s\}_{(E,\Phi)}$ and $\{P^+_s\}_{(E,\Phi)}$ be the families of projectors defined by the set of solutions $\{N\}_{(E,\Phi)}$ for each admissible section $s \in \Gamma_S(X, Y)$ by formulas (3.8) and (3.9), and let $\{\widetilde{\Theta}\}_{(E,\Phi)}$ be the corresponding family of Cartan forms along $S^{(2)}$ defined by (4.1).

Proposition 1. *The families $\{P_s\}_{(E,\Phi)}$, $\{P^+_s\}_{(E,\Phi)}$ and $\{\widetilde{\Theta}\}_{(E,\Phi)}$ do not depend on the chosen vector bundle E nor on the bundle morphism $\Phi : J^1Y \to E$ defining the constraint submanifold $S = \Phi^{-1}(0_E)$.*

Proof. If $q' : E' \to Y$ and $\Phi' : J^1Y \to E'$ are another k-rank vector bundle and bundle morphism on Y satisfying Hypothesis (HY1), following the previous Lemma it suffices to prove that if P_s, P^+_s and $\widetilde{\Theta}$ are the corresponding projections and Cartan form along $S^{(2)}$ for a solution $N \in \{N\}_{(E,\Phi)}$ and P'_s, P'^+_s and $\widetilde{\Theta}'$ the corresponding ones for the solution $N' = \tau^{(2)}N \in \{N'\}_{(E',\Phi')}$ then: $P_s = P'_s$, $P^+_s = P'^+_s$ and $\widetilde{\Theta} = \widetilde{\Theta}'$.

To prove the identity $P_s = P'_s$, following (3.8), it suffices to see that $N_s \circ (j^1D^{\mathrm{v}}_s)\Phi = N'_s \circ (j^1D^{\mathrm{v}}_s)\Phi'$ for any $D^{\mathrm{v}}_s \in \Gamma(X, s^*VY)$, which is proven by the following local computation:

$$\begin{aligned}N'_s \circ (j^1D^{\mathrm{v}}_s)\Phi' &= N'_s \circ \mathrm{d}\Phi'(j^1D^{\mathrm{v}}_s) = N'^j_{\alpha'}\mathrm{d}\phi'^{\alpha'}(j^1D^{\mathrm{v}}_s)\frac{\partial}{\partial y^j} = b^{\alpha}_{\alpha'}N^j_\alpha a^{\alpha'}_{\beta}\mathrm{d}\phi^{\beta}(j^1D^{\mathrm{v}}_s)\frac{\partial}{\partial y^j} \\ &= b^{\alpha}_{\alpha'}a^{\alpha'}_{\beta}N^j_\alpha\mathrm{d}\phi^{\beta}(j^1D^{\mathrm{v}}_s)\frac{\partial}{\partial y^j} = N^j_\alpha\mathrm{d}\phi^{\alpha}(j^1D^{\mathrm{v}}_s)\frac{\partial}{\partial y^j} \\ &= N_s \circ \mathrm{d}\Phi(j^1D^{\mathrm{v}}_s) = N_s \circ (j^1D^{\mathrm{v}}_s)\Phi\end{aligned}$$

In a similar way, the equation $\mathcal{E}_s(N) \circ \Omega_{\Phi\omega} = \mathcal{E}_s(N') \circ \Omega_{\Phi'\omega}$ is proven and therefore following formula (3.10) and the identity $P_s = P'_s$ we conclude that also $P^+_s = P'^+_s$.

Finally, applying (4.1) and considering the identity $\mathcal{E}_{\mathcal{L}\omega}(N) \circ (\theta \bar{\wedge} \Omega_{\Phi\omega}) = \mathcal{E}_{\mathcal{L}\omega}(N') \circ (\theta \bar{\wedge} \Omega_{\Phi'\omega})$ along $S^{(2)}$ (which is proven with the same local computations as for the previous identities), we obtain $\widetilde{\Theta} = \widetilde{\Theta}'$ along $S^{(2)}$, thus concluding the proof. □

As a consequence of this proposition, the whole theory developed in Section 3 and the corresponding Cartan and Noether formulations in Section 4 do not depend on the chosen vector bundle E and bundle morphism $\Phi : J^1Y \to E$ that define the constraint submanifold $S = \Phi^{-1}(0_E)$.

Regarding the contents of Section 5, we may state the following: Given (E, Φ) and (E', Φ'), the vector bundle isomorphism $\tau : E_S \to E'_S$ induces an isomorphism between $S \times_Y E^* = E^*_S$ and $S \times_Y E'^* = E'^*_S$ that transforms one to the other both Cartan forms $\widehat{\Theta} = \Theta_{\mathcal{L}\omega} + \lambda \circ \Theta_{\Phi\omega}$ and $\widehat{\Theta}' = \Theta_{\mathcal{L}\omega} + \lambda' \circ \Theta_{\Phi'\omega}$ defined by formulae (5.2), and gives in this way a canonical isomorphism between both variational formulations on $S \times_Y E^*$ and $S \times_Y E'^*$ as described in Section 5. In particular, we must note the independence of the notion of regularity (Definition 5.3) with respect to the chosen pair (E, Φ) that defines the constraint submanifold.

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